THE PRINCIPLES OF

EQUATION SUB-ELEMENT THEORY

VOLUME ONE OF FOUR

SECTIONS 1 THRU 14

TRUE SCANS

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SECTION 1. *INTRODUCTION*

This treatise formally *establishes* the principles of ***Equation Sub-elements*** – being a *headlong excursion* into the topsy-turvy preoccupation of *classifying mathematical equation formats*.

Equation sub-elements, hereinafter deemed *RST terminology*, reveal just how *Quadratic and Cubic Equations* behave with respect to one another.

They operate from *behind the scenes*, governing equation *interaction* through a network of strict rules.

RST terminology acts to associate coefficient structures evident within algebraic equation formats to their very root sets; thereby enabling them to be directly solved through the use of newly presented formulas.

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SECTION 2. *FUNDAMENTAL INFORMATION.*

This section presents *fundamental information* that is to serve as a basis for upcoming theory presented in the remainder of this treatise.

2.1. Well-known Cubic Equations and an Introduction of Cubic Reduction.

2.1.1. *Well-known Cubic Equations..*

Known, or ***previously established*** *Cubic Equations* for the sine, cosine and tangent of *any given angle* of value 3θ° are as follows:

Equation 1. ***Known Cubic Equation for the Cosine (3θ).***

$$4\cos^3\theta - 3\cos\theta = \cos(3\theta) \quad \textit{FOOTNOTE1}$$

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta)$$

$$= \frac{3}{4}\cos\theta + \frac{\tau}{4}$$

Equation 2. ***Known Cubic Equation for the Sine (3θ).***

$$3\sin\theta - 4\sin^3\theta = \sin(3\theta) \quad \textit{FOOTNOTE 2}$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$$

$$= \frac{3}{4}\sin\theta - \frac{\eta}{4}$$

Equation 3. ***Known Cubic Equation for the Tangent (3θ).***

$$\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \quad \text{FOOTNOTE 3}$$

$$\tan^3\theta = 3\tan\theta - \tan(3\theta)(1 - 3\tan^2\theta)$$

$$= 3\tan\theta - \zeta(1 - 3\tan^2\theta)$$

1. CRC Standard Mathematical Tables Twelfth Edition; The Chemical Rubber Co. Cleveland, OH; Jan. 1964; pg. 408.
2. Ibid.
3. Ibid.

2.1.2. *An Introduction of Cubic Reduction.*

Cubic reduction is a method whereby equations of the third order, or higher, may be *transformed* into lower order *Linear* and *Quadratic Equations*.

Simply speaking, ***cubic reduction*** occurs whenever right-hand terms from either *Equation 1*, *Equation 2*, or *Equation 3* become substituted for respective left-hand *cubic terms* as they might appear in *other higher* order equations.

A first excursion into *cubic reduction* may best be served by means of examining the following *equality*:

$$\sin\phi = \frac{1}{2\cos\theta}$$

Multiplying both sides of the equation by a factor of $2\cos\theta$, otherwise known as ***cross multiplication*** yields:

$$2\sin\phi\cos\theta = 1$$

$$(2\sin\phi\cos\theta)^3 = (1)^3$$

$$2(2\sin\phi\cos\theta)^3 = 2$$

$$(4\sin^3\phi)(4\cos^3\theta) = 2$$

Since *Equation 2*, applies to *any* variable (not only θ), it is obvious that:

$$4\sin^3\phi = 3\sin\phi - \sin(3\phi)$$

$$= 3\sin\phi - \lambda$$

Also,

$$4\cos^3\theta = 3\cos\theta + \cos(3\theta) \qquad \textit{[Ref. Equation 1]}$$

$$= 3\cos\theta + \tau$$

Then by combining the above results,

$$(3\sin\phi - \lambda)(3\cos\theta + \tau) = 2$$

Substitution for $\sin\phi$ into the first term above yields:

$$\left(\frac{3}{2\cos\theta} - \lambda\right)(3\cos\theta + \tau) = 2$$

Or,

$$(3 - 2\lambda\cos\theta)(3\cos\theta + \tau) = 4\cos\theta$$

Then,

$$9\cos\theta + 3\tau - 6\lambda\cos^2\theta - 2\tau\lambda\cos\theta = 4\cos\theta$$

Whereby, the following results:

Equation 4. An *Equation 1 Reduction.*

$$\cos^2\theta + \left(\frac{2\tau\lambda - 5}{6\lambda}\right)\cos\theta - \frac{\tau}{2\lambda} = 0$$

The above result depicts a *transformation* of *Equation 1*. Since it includes only linear and second order terms of the cos θ, it truly represents a *reduced* form *of* Cubic *Equation 1*.

Notice that this equation is conveniently expressed in terms of

- Readily identifiable numerical coefficients
- A known, or given, cos(3θ) value, τ,
- And a second variable, unknown $\sin(3\phi)$ term, λ

The λ *variable* permits *Equation 4* to maintain and reflect different quantitative values for cos θ.

Equation 4 is formatted such that when cos θ *is irrational*, it nevertheless remains related, or linked via the *Quadratic Formula*, to a set of given, or known coefficients, including τ, via only one other *unknown quantity*, λ.

2.2. Quadratic vs. Complex Quadratic Equations.

This treatise attempts to establish a distinction between two categories of *Quadratic Equations* as follows:

- Those which *express* first and second order mathematical combinations of a singular *unknown quantity*, in this case, 'x' -- Hereinafter to be known as *Normal Quadratic Equations*, or just *Quadratic Equations*.

- Those which *express* first and second order combinations of multiple unknown quantities, in this case, 'x_1', 'x_2', etc. -- Hereinafter to be known as 'so-called' ***Complex Quadratic Equations***.

Quadratic Equations represent *various combinations* of the following *three* types of like terms:

1) Those which are expressed in 'x^2';
2) Those which are expressed in 'x'; and
3) Those which are completely devoid of 'x' and 'x^2' terminology.

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Similar like terms may be collected together via the *Associative Law*, ultimately, to produce the following well known *Quadratic Equation* format, where a, b, and c depict given coefficients:

$$ax^2 + bx + c = 0$$

The following well-known *Quadratic Formula* expresses 'x' in terms of its three known coefficients, a, b, and c:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The *Quadratic Formula* returns two distinct *numerical* roots which satisfy any *Quadratic Equation* of the above form it becomes applied to.

Conversely, when the *Quadratic Formula* is applied to a *Complex Quadratic Equation*, it produces *root sets* where the value of the first root is ascribed in terms of its other, yet to be resolved unknown. Hence, a quantifiable value cannot be realized until such time that the second unknown term becomes *independently* ascertained.

A simple reformatting of the *Complex Quadratic Equation* shown below demonstrates this:

$$ax_1^{\ 2} + bx_1 + c + dx_2^{\ 2} + ex_2 + f = 0$$

Where,

- a, b, c, d, e, and f *represent given coefficients*
- 'x_1' and 'x_2' *represent unknown quantities*

Letting $g = c + dx_2^{\ 2} + ex_2 + f$ renders,

$$ax_1^{\ 2} + bx_1 + g = 0$$

Or via *Quadratic Formula*,

$$x_{1_a}; x_{1b} = \frac{-b \pm \sqrt{b^2 - 4ag}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4a(c + dx_2^{\ 2} + ex_2 + f)}}{2a}$$

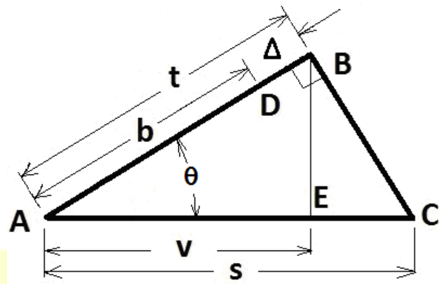
Above, the $x_{1_a}; x_{1b}$ *root set* is expressed in terms of *given coefficients* along with a radical which contains first and second order unknown x_2 terms.

2.3. Quadratic Equation Mapping

Figure 1 portrays a right triangle ABC, where line BE represents an altitude, and:

The length of side AB is designated as t
The length of side AC is designated as s
The length AE is designated as v
The angle CAB is designated as θ

Figure 1. Illustration of Elements Exhibited in the *Quadratic Formula*.



Then,

$$\cos\theta = \frac{t}{s} = \frac{v}{t}$$

Cross multiplying yields,

$$vs = t^2$$

$$v = \frac{t^2}{s}$$

Now, length AD is to be designated as b such that point D resides somewhere on line AB with b being smaller than length AB. Furthermore, length t represents the following:

$$t = \sqrt{b^2 - 4ac}$$

Such that,

$$t^2 = b^2 - 4ac$$

Or,

$$v = \frac{t^2}{s} = \frac{b^2 - 4ac}{s}$$

Delta is defined as follows:

$$\Delta = t - b = \overline{BD}$$

With respect to the *Quadratic Formula* presented below, a, b, and c meet the following constraints:

- a > 0
- b > 0
- c ≤ 0

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$$x_1; x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm t}{2a}$$

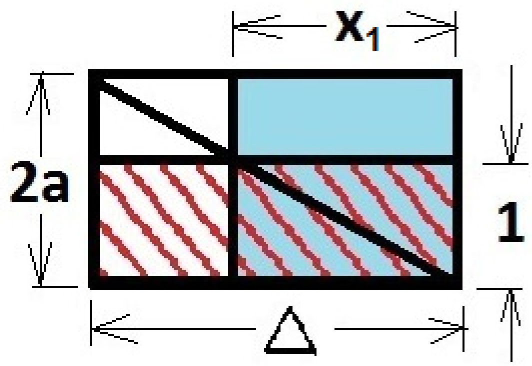
$$= \frac{\Delta}{2a}; \frac{-(b+t)}{2a}$$

Or,

$$\frac{x_1}{1} = \frac{\Delta}{2a}$$

A rectangle with adjacent sides equal to 2a and Δ, respectively, is portrayed in *Figure 2*.

Figure 2. Quadratic Equation Euclidean Mapping.



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Next, a diagonal which connects opposite corners of this rectangle is drawn.

Lastly, a horizontal line is inserted within the rectangle which describes the locus of points where its height is equal to one unit.

Per *Figure 2*, Δ/2a depicts the ratio of the respective sides of a right triangle whose hypotenuse is represented by the diagonal to the rectangle.

The intersection point of the rectangle’s diagonal with the line of height equal to one unit describes the vertex of a *second right triangle* also appearing in this figure.

Since these two above mentioned *right triangles* exhibit a common, equal angle between them, they are similar and must have respective sides meeting the ratio afforded above.

Hence, the horizontal length running from this newly identified vertex to the right side of the rectangle must be equal to x_1 in length.

Therefore, it is demonstrated that x may be <u>constructed</u> merely by applying the *coefficients* a, b, and c appearing in any given *Quadratic Formula*, based upon a calculation of the following two parameters:

- $t = \sqrt{b^2 - 4ac}$
- $\Delta = t - b$

2.4. Three Roots Does a Cubic Make!

Equations 1, 2, and 3 may be reconstituted as:

$$\tau = \cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$
$$\eta = \sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$
$$\zeta = \tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Now, since

$$\begin{aligned} 3\theta &= 3[\theta] \\ &= 3(\theta + 120^\circ) = 3\theta + 360^\circ = 3\theta \\ &= 3(\theta + 240^\circ) = 3\theta + 720^\circ = 3\theta \end{aligned}$$

It follows that

$$\begin{aligned} \tau &= \cos 3(\theta) &&= 4\cos^3\theta - 3\cos\theta \\ &= \cos[3(\theta + 120^o)] &&= 4\cos^3(\theta + 120^o) - 3\cos(\theta + 120^o) \\ &= \cos[(3(\theta + 240^o)] &&= 4\cos^3(\theta + 240^o) - 3\cos(\theta + 240^o) \end{aligned}$$

$$\begin{aligned} \eta &= \sin 3(\theta) &&= 3\sin\theta - 4\sin^3\theta \\ &= \sin[(3(\theta + 120^o)] &&= 3\sin(\theta + 120^o) - 4\sin^3(\theta + 120^o) \\ &= \sin[(3(\theta + 240^o)] &&= 3\sin(\theta + 240^o) - 4\sin^3(\theta + 240^o) \end{aligned}$$

$$\begin{aligned} \zeta &= \tan 3(\theta) &&= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \\ &= \tan[(3(\theta + 120^o)] &&= \frac{3\tan(\theta + 120^o) - \tan^3(\theta + 120^o)}{1 - 3\tan^2(\theta + 120^o)} \\ &= \tan[(3(\theta + 240^o)] &&= \frac{3\tan(\theta + 240^o) - \tan^3(\theta + 240^o)}{1 - 3\tan^2(\theta + 240^o)} \end{aligned}$$

Above, *three unique cosine values* produce the same value for τ, as represented in *Equation 1*. Hence, the *three roots* for the *Equation 1 cubic* are cos θ, cos (θ + 120°), and cos (θ + 240°), respectively.

Likewise, the three roots for *Equation 2* are sin θ, sin (θ + 120°), and sin (θ + 240°), respectively.

Lastly, the three roots for *Equation 3* are tan θ, tan (θ + 120°), and tan (θ + 240°), respectively.

It makes sense that cos θ is a root for *Equation 1* since the latter represents a *trisection equation* for any given value of cos (3θ), or τ, where (3θ)/3 = θ. However, the other two roots are not obvious. Hence, these assertions are validated below.

2.4.1. *The Cosine Cubic.*

Letting x_1, x_2, and x_3, respectively, represent the unique combination of functions expressed below:

$$x_1 = \cos\theta = \cos\theta_1$$
$$x_2 = \cos(\theta + 120^\circ) = \cos\theta_2 = -1/2\cos\theta - \sqrt{3}/2\sin\theta$$
$$x_3 = \cos(\theta + 240^\circ) = \cos\theta_3 = -1/2\cos\theta + \sqrt{3}/2\sin\theta$$

Then,

$$\begin{aligned}(x_1)(x_2)(x_3) &= \cos\theta[1/4\cos^2\theta - ¾\sin^2\theta]\\ &= (\cos\theta)[\cos^2\theta - 3\sin^2\theta]/4\\ &= (\cos\theta)[\cos^2\theta - 3(1-\cos^2\theta]/4\\ &= (\cos\theta)[4\cos^2\theta - 3]/4\\ &= (1/4)(4\cos^3\theta - 3\cos\theta)\\ &= \frac{1}{4}\cos(3\theta) \qquad \textit{[Ref. Equation 1]}\end{aligned}$$

Or,

$$x_1 x_2 x_3 = \frac{\tau}{4}$$

Also, by inspection:

$$x_1 + x_2 + x_3 = 0$$

And,

$$\begin{aligned}x_1x_2 + x_1x_3 + x_2x_3 &= x_1(x_2 + x_3) + x_2x_3\\ &= x_1(-x_1) + x_2x_3\\ &= \cos\theta(-\cos\theta) + (\cos^2\theta - 3\sin^2\theta)/4\\ &= -\cos^2\theta + [\cos^2\theta - 3(1-\cos^2\theta)]/4\\ &= -\cos^2\theta + [\cos^2\theta - 3/4]\\ &= -3/4\end{aligned}$$

Furthermore, a *Cubic Equation* in 'x' contains *three roots*, 'x_1', 'x_2' 'x_3' which must satisfy that equation. Then, the following three equalities must hold:

$$x = x_1$$

$$x = x_2$$

$$x = x_3$$

Or,

$$x - x_1 = 0$$

$$x - x_2 = 0$$

$$x - x_3 = 0$$

Accordingly, the product of the above must equal zero, as follows:

$$(x - x_1)(x - x_2)(x - x_3) = (x - x_1)[x^2 - x(x_2 + x_3) + x_2 x_3] = 0$$

$$= x^3 - x^2(x_1 + x_2 + x_3) + x(x_1 x_2 + x_1 x_3 + x_2 x_3) - x_1 x_2 x_3 = 0$$

Now, when x = cos θ, the above equation assumes the following form:

$$\cos^3\theta - \cos^2\theta(x_1 + x_2 + x_3) + \cos\theta(x_1 x_2 + x_1 x_3 + x_2 x_3) - x_1 x_2 x_3 = 0$$

Lastly the three prior relationships which were determined above are substituted in -- Namely:

Equation 5. ***The Product of the Roots from Equation 1 Equals τ/4.***

$$x_1 x_2 x_3 = \frac{\tau}{4}$$

Equation 6. ***The Summation of the Roots from Equation 1 Equals Zero.***

$$x_1 + x_2 + x_3 = 0$$

Equation 7. ***The Sum of Paired Products from Equation 1 Equals Minus ¾.***

$$x_1 x_2 + x_1 x_3 + x_2 x_3 = -\frac{3}{4}$$

This establishes the final form for the *Cubic Equation* as follows:

$$\cos^3\theta - \cos^2\theta(0) + \cos\theta(-3/4) - \tau/4 = 0$$

This is identical to *Equation 1*:

$$\cos^3\theta - (3/4)\cos\theta - \tau/4 = 0$$ *[Ref. Equation 1]*

Such identity proves that the three roots for *Equation 1* are indeed:

$$x_1 = cos\theta$$

$$x_2 = \cos(\theta + 120^o)$$

$$x_3 = \cos(\theta + 240^\circ)$$

2.4.2. *The Sine Cubic.*

Letting y_1, y_2, and y_3, respectively, represent the unique combination of functions expressed below:

y_1 = sin θ = sin θ_1

y_2 = sin (θ + 120°) = sin θ_2= −1/2 sin θ + $\sqrt{3}$/2 cos θ

y_3 = sin (θ + 240°) = sin θ_3= −1/2 sin θ − $\sqrt{3}$/2 cos θ

Then,

$$\begin{aligned}(y_1)(y_2)(y_3) &= sin\theta[1/4\sin^2\theta - ¾\cos^2\theta] \\ &= (sin\theta)[\sin^2\theta - 3\cos^2\theta]/4 \\ &= (\sin\theta)[\sin^2\theta - 3(1 - \sin^2\theta]/4 \\ &= (\sin\theta)[4\sin^2\theta - 3]/4 \\ &= (1/4)(4\sin^3\theta - 3\sin\theta) \\ &= -\frac{1}{4}\sin(3\theta)\end{aligned}$$

[Ref. Equation 2]

Or,

$$y_1 y_2 y_3 = -\frac{\eta}{4}$$

Also, by inspection:

$$y_1 + y_2 + y_3 = 0$$

And,

$$\begin{aligned}y_1 y_2 + y_1 y_3 + y_2 y_3 &= y_1(y_2 + y_3) + y_2 y_3 \\ &= y_1(-y_1) + y_2 y_3 \\ &= \sin\theta(-\sin\theta) + (\sin^2\theta - 3\cos^2\theta)/4 \\ &= -\sin^2\theta + [\sin^2\theta - 3(1 - \sin^2\theta)]/4 \\ &= -\sin^2\theta + [sin^2\theta - 3/4] \\ &= -3/4\end{aligned}$$

Furthermore, a *Cubic Equation* in 'y' contains *three roots*, 'y_1', 'y_2' 'y_3' which must satisfy that equation. Then, the following three equalities must hold:

$$y = y_1$$
$$y = y_2$$
$$y = y_3$$

Or,

$$y - y_1 = 0$$
$$y - y_2 = 0$$
$$y - y_3 = 0$$

Accordingly, the product of the above must equal zero, as follows:

$$(y-y_1)(y-y_2)(y-y_3) = (y-y_1)[y^2 - y(y_2+y_3) + y_2y_3] = 0$$
$$= y^3 - y^2(y_1+y_2+y_3) + y(y_1y_2 + y_1y_3 + y_2y_3) - y_1y_2y_3 = 0$$

Now, when y = sin θ, the above equation assumes the following form:

$$\sin^3\theta - \sin^2\theta(y_1+y_2+y_3) + \sin\theta(y_1y_2 + y_1y_3 + y_2y_3) - y_1y_2y_3 = 0$$

Lastly the three prior relationships which were determined above are substituted in -- Namely:

Equation 8. ***The Product of the Roots from Equation 2 Equals η/4.***

$$y_1y_2y_3 = -\frac{\eta}{4}$$

Equation 9. ***The Summation of the Roots from Equation 2 Equals Zero.***

$$y_1 + y_2 + y_3 = 0$$

Equation 10. ***The Sum of Paired Products from Equation 2 Equals Minus ¾.***

$$y_1y_2 + y_1\,y_3 + y_2\,y_3 = -3/4$$

This establishes the final form for the *Cubic Equation* as follows:

$$\sin^3\theta - \sin^2\theta(0) + \sin\theta(-3/4) + \eta/4 = 0$$

This is identical to *Equation 2*:

$$\sin^3\theta - (3/4)\sin\theta + \eta/4 = 0$$ *[Ref. Equation 2]*

Such identity proves that the three roots for *Equation 2* are indeed:

$$y_1 = \sin\theta$$
$$y_2 = \sin(\theta + 120^o)$$
$$y_3 = sin(\theta + 240^o)$$

2.4.3. *The Tangent Cubic.*

Letting z_1, z_2, and z_3, respectively, represent the unique combination of functions expressed below:

$$z_1 = \tan\theta = \tan\theta_1$$

$$z_2 = \tan(\theta + 120°) = \tan\theta_2 = \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta}$$

$$z_3 = \tan(\theta + 240°) = \tan\theta_3 = \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$$

Then,

$$(z_1)(z_2)(z_3) = \tan\theta\left[\frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}\right]$$

$$= \frac{\tan^3\theta - 3\tan\theta}{1 - 3\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan(3\theta)(1 - 3\tan^2\theta) - 3\tan\theta}{1 - 3\tan^2\theta} \quad \text{[Ref. Equation 3]}$$

$$= \frac{-\tan(3\theta)(1 - 3\tan^2\theta)}{1 - 3\tan^2\theta}$$

$$= -\tan(3\theta)$$

Or,

$$z_1 z_2 z_3 = -\zeta$$

Also:

$$z_1 + z_2 + z_3 = \frac{(\tan\theta)[1 - 3\tan^2\theta] + [\tan\theta - \sqrt{3}][1 - \sqrt{3}\tan\theta] + [\tan\theta + \sqrt{3}][1 + \sqrt{3}\tan\theta]}{1 - 3\tan^2\theta}$$

$$= \frac{\tan\theta - 3\tan^3\theta + \tan\theta - \sqrt{3}\tan^2\theta - \sqrt{3} + 3\tan\theta + \tan\theta + \sqrt{3}\tan^2\theta + \sqrt{3} + 3\tan\theta}{1 - 3\tan^2\theta}$$

$$= \frac{9\tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \frac{9\tan\theta - 3[3\tan\theta - \tan(3\theta)(1 - 3\tan^2\theta)]}{1 - 3\tan^2\theta} \quad \text{[Ref. Equation 3]}$$

$$= 3\tan(3\theta)$$

$$= 3\zeta$$

Then,

$$z_2 + z_3 = 3\zeta - z_1$$

$$= 3\zeta - \tan\theta$$

So,

$$
\begin{aligned}
z_1z_2 + z_1z_3 + z_2z_3 &= z_1(z_2 + z_3) + z_2z_3 \\
&= \tan\theta(3\zeta - \tan\theta) + \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}\left[\frac{\tan\theta}{\tan\theta}\right] \\
&= \tan\theta(3\zeta - \tan\theta) - \frac{3\tan\theta - \tan^3\theta}{(1 - 3\tan^2\theta)(\tan\theta)} \\
&= \tan\theta(3\zeta - \tan\theta) - \zeta / \tan\theta \\
&= \frac{\tan^2\theta(3\zeta - \tan\theta) - \zeta}{\tan\theta} \\
&= \frac{-\tan^3\theta - \zeta(1 - 3\tan^2\theta)}{\tan\theta} \\
&= \frac{-\tan^3\theta - (3\tan\theta - \tan^3\theta)}{\tan\theta} \quad \text{[Ref. Equation 3]} \\
&= -3
\end{aligned}
$$

Furthermore, a *Cubic Equation* in 'z' contains three roots, 'z_1', 'z_2' 'z_3' which must satisfy that equation. Then, the following three equalities must hold:

$$z = z_1$$
$$z = z_2$$
$$z = z_3$$

Or,

$$z - z_1 = 0$$
$$z - z_2 = 0$$
$$z - z_3 = 0$$

Accordingly, the product of the above must equal zero, as follows:

$$
\begin{aligned}
(z - z_1)(z - z_2)(z - z_3) &= (z - z_1)[z^2 - z(z_2 + z_3) + z_2z_3] \\
&= z^3 - z^2(z_1 + z_2 + z_3) + z(z_1z_2 + z_1z_3 + z_2z_3) - z_1z_2z_3 = 0
\end{aligned}
$$

Now, when z = tan θ, the above equation becomes:

$$\tan^3\theta - \tan^2\theta(z_1 + z_2 + z_3) + \tan\theta(z_1z_2 + z_1z_3 + z_2z_3) - z_1z_2z_3 = 0$$

Lastly the three prior relationships which were determined above are substituted in -- Namely:

Equation 11. ***The Product of the Roots from Equation 3 Equals Minus ζ.***

$$z_1z_2z_3 = -\zeta$$

Equation 12. ***The Summation of the Roots from Equation 3 Equals 3ζ.***

$$z_1 + z_2 + z_3 = 3\zeta$$

Equation 13. ***The Sum of Paired Products from Equation 3 Equals Minus 3.***

$$z_1z_2 + z_1z_3 + z_2z_3 = -3$$

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This establishes the final form for the *Cubic Equation* as follows:

$$\tan^3\theta - \tan^2\theta(3\zeta) + \tan\theta(-3) + \zeta = 0$$

This is identical to *Equation 3*:

$$\tan^3\theta - 3\zeta\tan^2\theta - 3\tan\theta + \zeta = 0 \qquad \textit{[Ref. Equation 3]}$$

Since these two equations appearing above are virtually identical, it proves that the three roots for *Equation 3* are:

$$z_1 = \tan\theta$$

$$z_2 = \tan(\theta + 120^o)$$

$$z_3 = \tan(\theta + 240^o)$$

2.5. The Cubic Correlation.

A reconstitution of *Equations 5 thru 13* is given below where right-hand side equation elements represent either *given trigonometric values for 3θ* or *rational numbers*, and left-hand side equation members denote respective *Equation 1, Equation 2, and Equation 3 root combinations* which *they are comprised of*:

$$x_1x_2x_3 = \frac{\tau}{4}$$

$$y_1y_2y_3 = -\frac{\eta}{4}$$

$$z_1z_2z_3 = -\zeta$$

$$x_1 + x_2 + x_3 = 0$$

$$y_1 + y_2 + y_3 = 0$$

$$z_1 + z_2 + z_3 = 3\zeta$$

$$x_1x_2 + x_1x_3 + x_2x_3 = -3/4$$

$$y_1y_2 + y_1y_3 + y_2y_3 = -3/4$$

$$z_1z_2 + z_1z_3 + z_2z_3 = -3$$

Table 1 validates these *Equation 5 thru 13* relationships for 3θ = 60°.

TRUE SCANS

Table 1. Cubic Equation Root Correlation for 3θ = 60°.

3θ Deg	θ Deg	τ cos (3θ)	τ/4	x_1 cos θ	x_2 cos (θ+120°)	x_3 cos (θ+240°)	$x_1x_2x_3$ [τ/4]	$x_1+x_2+x_3$ [0]	$x_1x_2+x_1x_3+x_2x_3$ [-¾]
60	20	0.500000	0.125000	0.939692620785908	-0.76604444311898	-0.17364817766693	0.125000	0.000000	-0.75

3θ Deg.	θ Deg.	η sin (3θ)	η/4	y_1 sin θ	y_2 sin (θ+120°)	y_3 sin (θ+240°)	$y_1y_2y_3$ [-η/4]	$y_1+y_2+y_3$ [0]	$y_1y_2+y_1y_3+y_2y_3$ [-¾]
60	20	0.866025404	0.216506351	0.342020143325669	0.64278760968654	-0.98480775301221	-0.216506351	0.000000	-0.75

3θ Deg.	θ Deg.	ζ tan (3θ)	3ζ	z_1 tan θ	z_2 tan (θ+120°)	z_3 tan (θ+240°)	$z_1z_2z_3$ [-ζ]	$z_1+z_2+z_3$ [3ζ]	$z_1z_2+z_1z_3+z_2z_3$ [-3]
60	20	1.732050808	5.196152423	0.363970234266202	-0.83909963117728	5.67128181961771	-1.732050808	5.196152423	-3

SECTION 3. *COMPLEX QUADRATIC EQUATIONS.*

This section develops new sets of *Complex Quadratic Equations* and introduces proposed supporting theory which explains how the new relationships should apply.

It also charts relevant relationships in order to further characterize and demonstrate them.

3.1. Development and Proposed Supporting Theory.

Where,

$$z_3 = 3\zeta - (z_1 + z_2) \qquad \textit{[Ref. Equation 12]}$$

$$\begin{aligned} z_1 z_2 z_3 &= z_1 z_2 [3\zeta - (z_1 + z_2)] \\ &= (3\zeta) z_1 z_2 - z_1 z_2 (z_1 + z_2) \\ &= -\zeta \qquad \textit{[Ref. Equation 11]} \end{aligned}$$

Therefore, the following equation becomes established:

Equation 14. ***A Complex Quadratic Equation that Relates 'z_1', 'z_2', and 'ζ'.***

$$z_1^2 z_2 + z_1 z_2^2 - (3\zeta) z_1 z_2 - \zeta = 0$$

Notice that the above equation *expresses only linear and second order terms* in both *of its two unknown components*, z_1 and z_2. Hence it qualifies as a *Complex Quadratic Equation*.

Dividing through by z_2 yields

$$z_1^2 + [\frac{z_2^2 - (3\zeta) z_2}{z_2}] z_1 - \frac{\zeta}{z_2} = 0$$

Likewise, dividing through by z_1 yields

$$z_2^2 + [\frac{z_1^2 - (3\zeta) z_1}{z_1}] z_2 - \frac{\zeta}{z_1} = 0$$

Simplifying second terms in each of the above two equations gives the following results:

$$z_1^2 + (z_2 - 3\zeta) z_1 - \frac{\zeta}{z_2} = 0$$

$$z_2^2 + (z_1 - 3\zeta) z_2 - \frac{\zeta}{z_1} = 0$$

Note, that these equations may become even further simplified by substituting *ascertained values* for unknowns exhibited in the denominators of each of the respective third terms above. Then, such *Complex Quadratic Equations* would become *Normal Quadratic Equations*.

Quantifications for each of the respective third terms above are established as follows, where:

$$\zeta(1-3\tan^2\theta) = 3\tan\theta - \tan^3\theta$$ *[Ref. Equation 3]*

$$\zeta - 3\zeta\tan^2\theta = 3\tan\theta - \tan^3\theta$$

$$\frac{\zeta}{\tan\theta} = 3\zeta\tan\theta + 3\tan\theta - \tan^2\theta$$

Or,

$$\frac{\zeta}{\tan\theta_1} = 3\zeta\tan\theta_1 + 3 - \tan^2\theta_1$$

$$\frac{\zeta}{\tan\theta_2} = 3\zeta\tan\theta_2 + 3 - \tan^2\theta_2$$

$$\frac{\zeta}{\tan\theta_3} = 3\zeta\tan\theta_3 + 3 - \tan^2\theta_3$$

Then it follows that,

$$\frac{\zeta}{z_1} = 3\zeta z_1 + 3 - z_1{}^2$$

$$\frac{\zeta}{z_2} = 3\zeta z_2 + 3 - z_2{}^2$$

$$\frac{\zeta}{z_3} = 3\zeta z_3 + 3 - z_3{}^2$$

So, substituting these respective third term derivations into the above *Complex Quadratic Equation* development yields the following results:

$$z_1{}^2 + (z_2 - 3\zeta)z_1 - (3\zeta z_2 + 3 - z_2{}^2) = 0$$

$$z_2{}^2 + (z_1 - 3\zeta)z_2 - (3\zeta z_1 + 3 - z_1{}^2) = 0$$

Applying the *Quadratic Formula* shown below produces:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z_1 = \frac{-(z_2 - 3\zeta) \pm \sqrt{(z_2 - 3\zeta)^2 - 4(1)[-(3\zeta z_2 + 3 - z_2{}^2)]}}{2(1)}$$

$$z_2 = \frac{-(z_1 - 3\zeta) \pm \sqrt{(z_1 - 3\zeta)^2 - 4(1)[-(3\zeta z_1 + 3 - z_1{}^2)]}}{2(1)}$$

Expanding terms renders:

$$z_1 = \frac{(3\zeta - z_2) \pm \sqrt{(z_2{}^2 - 6\zeta z_2 + 9\zeta^2) + (12\zeta z_2 + 12 - 4z_2{}^2)}}{2(1)}$$

$$z_2 = \frac{(3\zeta - z_1) \pm \sqrt{(z_1{}^2 - 6\zeta z_1 + 9\zeta^2) + (12\zeta z_1 + 12 - 4z_1{}^2)}}{2(1)}$$

Combining terms results in:

Equation 15. ***An Expression for the 'z_1' Root for Equation 14.***

$$z_1 = \frac{(3\zeta - z_2) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_2 - 3z_2{}^2}}{2}$$

Equation 16. ***An Expression for the 'z_2' Root for Equation 14.***

$$z_2 = \frac{(3\zeta - z_1) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_1 - 3z_1{}^2}}{2}$$

Both of the above equations assume exactly the same structure. This is easily evidenced by means of swapping locations for each variable.

The above two equations, relate z_1 and z_2 unknowns to a known tan (3θ), or ζ, value.

Either z_1 or z_2 may be solved for, once given the value of its constituent, remaining unknown. For any given value of ζ, a distinct family of unknowns becomes characterized as the first unknown assumes various, arbitrary values. This emphasizes the fact that both of the above two equations may be viewed, or considered as representing *distinct families of root sets*.

Now, for the particular condition when $\zeta = \sqrt{3}$, *Complex Quadratic Equation 14* portrays the two unknown quantities, z_1 and z_2, as follows:

Equation 17. ***Expression for Equation 14 when ζ = √3.***

$$z_1{}^2 z_2 + z_1 z_2{}^2 - 3\sqrt{3} z_1 z_2 - \sqrt{3} = 0$$

Since *Complex Quadratic Equations* constitute ***families of unknowns***, or root sets, resultant z_2 values may be determined, or generated, once arbitrary z_1 values are input into the above equation; and vice versa.

However, even though meeting the criterion of this above, given equation, selected roots sets may not meet additional criteria which may satisfactorily relate them to $\zeta = \sqrt{3}$.

So, for the specific condition when $z_2 = 1$:

$$z_1{}^2 z_2 + z_1 z_2{}^2 - 3\sqrt{3} z_1 z_2 - \sqrt{3} = 0 \quad \textit{[Ref. Equation 17]}$$

$$z_1{}^2(1) + z_1(1)^2 - 3\sqrt{3} z_1(1) - \sqrt{3} = 0$$

Then,

Equation 18. ***Expression for Equation 17 when 'z_2' = 1.***

$$z_1{}^2 + (1 - 3\sqrt{3}) z_1 - \sqrt{3} = 0$$

3.2. Verification.

With ζ held constant at a specified value of 0.6, the *first tabulation* below indicates that *Equation 15* calculates *different root set values* for each *unique z_2 value* that is applied to it.

The *second tabulation* below demonstrates that *Equation 15* and *Equation 16* are *inextricably linked* by virtue of the fact that they *share* multiple root values of 2.685182821, and -1.347582821, respectively. As indicated, such relationship holds when applying a z_1 root obtained from *Equation 15* into *Equation 16*.

Notice that the left-hand sides of the two tabulations below are exactly the same. This is because in both cases *Equation 15* is solving for z_2 = 0.4624 at $\zeta = 0.6$.

Further below, *Equation 15 and Equation 16* are verified for accuracy when $\theta = 20^o$. Then,

z_1 = tan θ = tan 20° = 0.363970234
z_2 = tan (θ + 120°) = tan 140° = -tan(2θ) = -0.839099631
z_3 = tan (θ + 240°) = tan 260° = tan(4θ) = 5.67128182

Where:

$$\begin{aligned}\zeta &= \tan(3\theta) \\ &= \tan(3x20^o) \\ &= \tan 60^o \\ &= \sqrt{3}\end{aligned}$$

$$z_1 = \frac{(3\zeta - z_2) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_2 - 3z_2{}^2}}{2} \quad \textit{[Ref. Equation 15]}$$

$$z_2 = \frac{(3\zeta - z_1) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_1 - 3z_1{}^2}}{2} \quad \textit{[Ref. Equation 16]}$$

When $\zeta = 0.6$, and **z_2 = 0.4624**

$$z_1 = \frac{(3\zeta - z_2) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_2 - 3{z_2}^2}}{2}$$ [Ref. Equation 15]

$$z_1 = \frac{[3(0.6) - 0.4624] \pm \sqrt{(9(0.6)^2 + 12) + 6(0.6)(0.4624) - 3(0.4624)^2}}{2}$$

$$z_1 = \frac{1.3376 \pm \sqrt{(15.24) + (1.66464) - (0.64144128)}}{2}$$

$$z_1 = \frac{1.3376 \pm \sqrt{(16.26319872)}}{2}$$

$$z_1 = \frac{1.3376 \pm 4.032765642}{2} \quad z_1 = \tan\theta, \tan(4\theta)$$

z_1 = 2.685182821, -1.347582821

When $\zeta = 0.6$, and **z_2 = 0.5206**

$$z_1 = \frac{(3\zeta - z_2) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_2 - 3{z_2}^2}}{2}$$ [Ref. Equation 15]

$$z_1 = \frac{[3(0.6) - 0.5206] \pm \sqrt{(9(0.6)^2 + 12) + 6(0.6)(0.5206) - 3(0.5206)^2}}{2}$$

$$z_1 = \frac{1.2794 \pm \sqrt{(15.24) + (1.87416) - (0.81307308)}}{2}$$

$$z_1 = \frac{1.2794 \pm \sqrt{(16.30108692)}}{2}$$

$$z_1 = \frac{1.2794 \pm 4.037460454}{2} \quad z_1 = \tan\theta, \tan(4\theta)$$

z_1 = 2.658430227, -1.379030227

When $\zeta = 0.6$, and **z_2 = 0.4624**

$$z_1 = \frac{(3\zeta - z_2) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_2 - 3{z_2}^2}}{2}$$ [Ref. Equation 15]

$$z_1 = \frac{3(0.6) - 0.4624 \pm \sqrt{9(0.6)^2 + 12 + 6(0.6)(0.4624) - 3(0.4624)^2}}{2}$$

$$z_1 = \frac{1.3376 \pm \sqrt{(15.24) + (1.66464) - (0.64144128)}}{2}$$

$$z_1 = \frac{1.3376 \pm \sqrt{(16.26319872)}}{2}$$

$$z_1 = \frac{1.3376 \pm 4.032765642}{2}$$

z_1 = 2.685182821, -1.347582821

But, when $\zeta = 0.6$, and z_1 = 2.68518281

$$z_2 = \frac{(3\zeta - z_1) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_1 - 3{z_1}^2}}{2}$$ [Ref. Equation 16]

$$z_2 = \frac{3(0.6) - 2.68518282 \pm \sqrt{9(0.6)^2 + 12 + 6(0.6)(2.68518282) - 3(2.68518282)^2}}{2}$$

$$z_2 = \frac{-0.88518282 \pm \sqrt{(15.24) + (9.666658152) - (21.63062033)}}{2}$$

$$z_2 = \frac{-0.88518282 \pm \sqrt{(3.276037822)}}{2}$$

$$z_2 = \frac{-0.88518282 \pm 1.809982824}{2}$$

z_1 = +0.462400002, -1.347582821

$$z_1 = \frac{3\sqrt{3} - (\text{-}0.839099631) \pm \sqrt{(9)(\sqrt{3})^2 + 12) + 6\sqrt{3}(\text{-}0.839099631) - 3(\text{-}0.839099631)^2}}{2}$$

$$= \frac{3\sqrt{3} - (0.363970234) \pm \sqrt{(9)(\sqrt{3})^2 + 12) + 6(\sqrt{3})(0.363970234) - 3(0.363970234)^2}}{2}$$

$$= \frac{3\sqrt{3} - (\text{-}0.839099631) \pm \sqrt{(39) + 6(1.732050808)(\text{-}0.839099631) - 3(\text{-}0.839099631)^2}}{2}$$

$$z_2 = \frac{3\sqrt{3} - (0.363970234) \pm \sqrt{(39) + 6(1.732050808)(0.363970234) - 3(0.363970234)^2}}{2}$$

Table 2. ***'z₁'and 'z₂' Root Determination Table.***

z_1 Root Determination	z_2 Root Determination
$z_1 = \frac{6.035252054 \pm \sqrt{(39) - 8.720179163 - 2}}{2}$	$z_2 = \frac{4.832182188 \pm \sqrt{(39) + 3.782489629 - (}}{2}$
$z_1 = \frac{6.035252054 \pm \sqrt{28.16755626}}{2}$	$z_2 = \frac{4.832182188 \pm \sqrt{42.38506663}}{2}$
$z_1 = \frac{6.035252054 \pm 5.307311585}{2}$	$z_2 = \frac{4.832182188 \pm 6.510381451}{2}$
$z_1 = +0.363970234, +5.67128182$	$z_2 = +5.67128182, \text{-}0.839099631$
$z_1 = \tan\theta, \tan(4\theta)$	$z_2 = \tan(4\theta), -\tan(2\theta)$

Hence, z_1 and z_2 results given in *table 2* match respective values for tan 20°, tan 140°, and tan 260° shown above, thereby demonstrating *Equation 15 and Equation 16* accuracy.

Now, with respect to *Equation 18*, '*Completing the Square*' gives the following roots:

$$z_1^{\,2} + (1 - 3\sqrt{3})z_1 - \sqrt{3} = 0 \quad \textit{[Ref. Equation 18]}$$

$$z_1^{\,2} + (1 - 3\sqrt{3})z_1 + [\frac{(1 - 3\sqrt{3})}{2}]^2 = [z_1 + \frac{(1 - 3\sqrt{3})}{2}]^2 = \sqrt{3} + [\frac{(1 - 3\sqrt{3})}{2}]^2$$

Taking the square root of each side renders:

$$z_1 + \frac{(1 - 3\sqrt{3})}{2} = \sqrt{\sqrt{3} + [\frac{(1 - 3\sqrt{3})}{2}]^2}$$

$$z_1 = -\frac{(1 - 3\sqrt{3})}{2} \pm \sqrt{\frac{4\sqrt{3} + (1 - 6\sqrt{3} + 27)}{4}}$$

$$= \frac{3\sqrt{3} - 1 \pm \sqrt{24.53589838}}{2}$$

$$= \frac{4.196152423 \pm 4.953372425}{2}$$

$$= 4.574762424, \text{-}\,0.378610001$$

TRUE SCANS

Checks:

$$z_1^{\,2} z_2 + z_1 z_2^{\,2} - 3\sqrt{3} z_1 z_2 - \sqrt{3} = 0 \qquad \text{[Ref. Equation 17]}$$

$$4.574762424^2 (1) + (4.574762424)(1)^2 - 3\sqrt{3}(4.574762424)(1) - \sqrt{3} = 0$$

$$20.92845124 + 4.574762424 - 23.77116286 - \sqrt{3} = 0$$

$$0 = 0$$

$$z_1^{\,2} z_2 + z_1 z_2^{\,2} - 3\sqrt{3} z_1 z_2 - \sqrt{3} = 0 \qquad \text{[Ref. Equation 17]}$$

$$(\text{-}0.378610001)^2 (1) + (\text{-}0.378610001)(1)^2 - 3\sqrt{3}(\text{-}0.378610001)(1) - \sqrt{3} = 0$$

$$0.143345532 - 0.378610001 + 1.967315274 - \sqrt{3} = 0$$

$$0 = 0$$

Validating *Equation 14* for a second value, when

$$\zeta = \sqrt{5} \quad = \tan 65.90515745^\circ = \tan (3\theta)$$

$$3\theta = 65.90515745^\circ$$

$$\theta = \frac{65.90515745^\circ}{3}$$

$$= 21.96838582^\circ$$

z_1 = tan θ = tan 21.96838582° = 0.403384527
z_2 = tan (θ + 120°) = tan 141.96838582° = -0.782174586
z_3 = tan (θ + 240°) = tan 261.96838582° = 7.086993995

Where,

$$z_2 = \frac{(3\zeta - z_1) \pm \sqrt{(9\zeta^2 + 12) + 6\zeta z_1 - 3z_1^{\,2}}}{2} \qquad \text{[Ref. Equation 16]}$$

$$z_2 = \frac{(3\sqrt{5} - 0.403384527) \pm \sqrt{[(9(5) + 12] + [6\sqrt{5}(0.403384527)] - [3(0.403384527)^2]}}{2}$$

$$= \frac{(6.304819406) \pm \sqrt{57 + 5.411971341 - 0.488157229}}{2}$$

$$= \frac{(6.304819406) \pm \sqrt{61.92381411}}{2}$$

$$= 7.086993992, -0.782174586$$

$$= \tan (\theta + 240^\circ), \tan (\theta + 120^\circ)$$

Then,

$$z_1^{\,2} z_2 + z_1 z_2^{\,2} - (3\zeta) z_1 z_2 - \zeta = 0 \qquad \text{[Ref. Equation 14]}$$

$$(0.403384527)^2 (\text{-}0.782174586) + (0.403384527)(\text{-}0.782174586)^2 - (3\sqrt{5})(0.403384527)(\text{-}0.782174586) - \sqrt{5} = 0$$

$$\text{-}\,0.127274726 + 0.246789476 + 2.116553221 - \sqrt{5} = 0$$

$$\sqrt{5} - \sqrt{5} = 0$$

As such *Equations 14 thru 16* are accurate too since z_2 *results* match the initial, respective z_2 and z_3 values posted just above.

3.3. Charting.

Table 3 presented below capitalizes on the fact that *Equation 17*, being a *Complex Quadratic Equation*, characterizes *dual root sets,* with each distinct z_1 value selection returning two solutions for z_2, and vice versa.

Below are furnished four distinct values for z_2, each of which are accompanied by associated z_1 dual root set values which were determined via *Equation* 17. Accordingly, each *root set* occupies two consecutive rows in *Table 3*:

1st root set,

$z_2 = 1$
$z_1 = 4.574762424, -0.378610001$

2nd root set,

$z_2 = \tan 20^\circ = 0.363970234$
$z_1 = -0.839099631, 5.67128182$

3rd root set,

$$z_2 = \tan(\theta + 120^o) = \tan 140^o = -\tan(2\theta) = -0.839099631$$

$z_1 = 0.363970234, 5.67128182$

Fourth root set,

$$z_2 = \tan(\theta + 120^o) = \tan 140^o = -\tan(2\theta) = -0.839099631$$

$z_1 = 0.363970234, 5.67128182$

The first two columns in *Table 3* itemize above respective root set z_1 and z_2 values, while columns three thru six itemize respective calculations for each of the respective terms contained in *Equation* 17. The last column of *Table 3* enumerates respective *summations* for each row.

Since roots represent selected values of z_1 and z_2 which satisfy *Equation* 17, they occur when last column summations equal zero. Since all *Table 3* last column summations equal zero, then all z_1 and z_2 values listed depict *true roots* of *Equation 19*.

Since *Equation 17* is symmetrical with respect to z_1 and z_2, root values are swapped in rows 3 and 5; rows 4 and 7; and in rows 6 and 8, respectively.

TRUE SCANS

Table 3. Four Root Sets For the *Complex Quadratic Equation 17.*

z_1	z_2	$z_1^2z_2$	$z_1z_2^2$	$-3\sqrt{3}z_1z_2$	$-\sqrt{3}$	$z_1^2z_2 + z_1z_2^2 - 3\sqrt{3}z_1z_2 - \sqrt{3} = 0$
4.574762424	1	20.92845124	4.574762424	-23.77116285	-1.732050808	0
-0.378610001	1	0.143345533	-0.378610001	1.967315274	-1.732050808	0
-0.839099631	0.363970234	0.256267144	-0.111159162	1.586942825	-1.732050808	0
5.67128182	0.363970234	11.70653387	0.751299266	-10.72578233	-1.732050808	0
0.363970234	-0.839099631	-0.111159162	0.256267144	1.586942825	-1.732050808	0
5.67128182	-0.839099631	-26.98832852	3.993082556	24.72729677	-1.732050808	0
0.363970234	5.67128182	0.751299266	11.70653387	-10.72578233	-1.732050808	0
-0.839099631	5.67128182	3.993082556	-26.98832852	24.72729677	-1.732050808	0

TRUE SCANS

3.4. Practical Application.

Certain well-*known equations* in *physics* constitute *Complex Quadratic Equations*.

3.4.1. *Well-known Complex Quadratic Equations in Physics.*

The well-known *Conservation of Linear Momentum* for two colliding particles is given below. No relativistic effects occur since each mass is unaffected by the impact.

Conservation of Linear Momentum Equation.

$$m_1V_1 + m_2V_2 = m_1v_1 + m_2v_2$$ Footnote 1

Where,

m_1 = Mass of 1st particle,
m_2 = Mass of 2nd particle,
V_1 = Velocity of 1st particle before collision,
V_2 = Velocity of 2nd particle before collision,
v_1 = Velocity of 1st particle after collision, and
v_2 = Velocity of 2nd particle after collision.

Its associated *Complex Quadratic Equation* is as follows:

$$m_1V_1 + m_2V_2 - m_1v_1 - m_2v_2 = 0$$

Secondly, for a *purely elastic condition*, the equation below represents the *Conservation of (Kinetic) Energy* for the same exact two colliding particles under the particular circumstance when their *Potential Energy* remains constant:

Conservation of (Kinetic) Energy Equation.

$$\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$ Footnote 2

Conversely, for a *non elastic condition*, the work, 'W' performed by the impact of these same two particles is the difference between right- and left-handed sides of the *Conservation of (Kinetic) Energy Equation*, as follows:

Work Performed Equation.

$$W = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 - (\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2)$$ Footnote 3

The *Coefficient of Restitution Equation* is afforded below. It represents the degree of particle inelasticity involved during the collision, where complete elasticity occurs when $\varepsilon = 1$, and total inelasticity applies when $\varepsilon = 0$.

The Coefficient of Restitution Equation.

$$\varepsilon = -\frac{v_1 - v_2}{V_1 - V_2}$$ Footnote 4

1. College Physics Third Edition; Sears & Zemansky; Addison Wesley Publishing Company, Reading, Massachusetts; 1960; page 159, Equation 8-4.
2. Ibid; page 129, Equation 7-5.
3. Ibid; page 129, Equation 7-6.
4. Ibid; page 163, Equation 8-5.

TRUE SCANS

3.4.2. *Complex Quadratic Equation Reformatting.*

The *Conservation of Linear Momentum Equation* above is mathematically operated upon as follows:

$$m_1V_1 + m_2V_2 = m_1v_1 + m_2v_2$$
$$m_1(V_1 - v_1) = m_2(v_2 - V_2)$$
$$m_1 = m_2\frac{v_2 - V_2}{V_1 - v_1}$$

By substituting this above m_1 expression into the *Work Performed Equation* below, the following reformat occurs:

$$\begin{aligned}W &= \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 - (\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2)\\
&= \frac{1}{2}m_1(V_1^2 - v_1^2) + \frac{1}{2}m_2(V_2^2 - v_2^2)\\
&= \frac{1}{2}(m_2\frac{v_2 - V_2}{V_1 - v_1})(V_1^2 - v_1^2) + \frac{1}{2}m_2(V_2^2 - v_2^2)\\
&= \frac{1}{2}m_2(\frac{v_2 - V_2}{V_1 - v_1})(V_1 - v_1)(V_1 + v_1) + \frac{1}{2}m_2(V_2 - v_2)(V_2 + v_2)\\
&= \frac{1}{2}m_2(v_2 - V_2)(V_1 + v_1) + \frac{1}{2}m_2(V_2 - v_2)(V_2 + v_2)\\
&= \frac{1}{2}m_2(V_2 - v_2)[(V_2 + v_2) - (V_1 + v_1)]\\
&= \frac{1}{2}m_2(V_2 - v_2)[(V_2 - V_1) + (v_2 - v_1)]\end{aligned}$$

From the *Coefficient of Restitution Equation*:

$$\begin{aligned}\varepsilon &= -\frac{v_1 - v_2}{V_1 - V_2}\\
\varepsilon(V_1 - V_2) &= -(v_1 - v_2)\\
&= v_2 - v_1\end{aligned}$$

Substituting this result into the above derivation yields the following:

$$\begin{aligned}W &= \frac{1}{2}m_2(V_2 - v_2)[(V_2 - V_1) + \varepsilon(V_1 - V_2)]\\
&= \frac{1}{2}m_2(V_2 - v_2)(V_2 - V_1)(1 - \varepsilon)\end{aligned}$$

Notice that for the *completely elastic condition* when ε is set equal to 1, the total work accomplished, or heat dissipated during the collision, equals zero. This means that the *Conservation of (Kinetic) Energy Equation* is maintained, where all particle velocity before the collision is translated into resultant velocities of the two respective particles after they collide.

Now, consider further that the two particles scrutinized in *Section 3.4.1* are *representative samples* from a batch of other *similar particles* that also are undergoing multiple collisions under similar laboratory conditions. Under this scenario, all particles also contain very nearly the same mass where, $m_1 = m_2 = m$.

For this condition, the equation derived above, then reduces to:

$$W = \frac{1}{2} m (V_2 - v_2)(V_2 - V_1)(1 - \varepsilon)$$

For any *given* mass, 'm', and *Coefficient of Restitution*, 'ε', the equation above personifies a ***Complex Quadratic Equation*** of three variables, V_1, V_2, and v_2, represented as follows:

$$(V_2 - v_2)(V_2 - V_1) = \frac{2W}{m(1-\varepsilon)}$$

$$(V_2 - v_2)(V_2 - V_1) - \frac{2W}{m(1-\varepsilon)} = 0$$

$$\boxed{V_2^{\,2} - (V_1 + v_2)V_2 + [V_1 v_2 - \frac{2W}{m(1-\varepsilon)}] = 0}$$

Once again, exercising the *Quadratic Formula* shown below produces:

$$V_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$V_2 = \frac{(V_1 + v_2) \pm \sqrt{(V_1 + v_2)^2 - 4(1)[V_1 v_2 - \frac{2W}{m(1-\varepsilon)}]}}{2(1)}$$

Consider a collision that occurs when V_1 is traveling in an opposite direction to V_2. When these velocity *magnitudes* are equal, then the following equation results:

$$W = \frac{1}{2} m (V_2 - v_2)(2V_2)(1 - \varepsilon)$$

Complex Quadratic Equation for Work Produced when Two Particles of Constant Mass, Velocity and Restitution Collide.

$$W - mV_2(V_2 - v_2)(1 - \varepsilon) = 0$$

Moreover, the two variables in the above equation may now be changed to:

V_2 = velocity before impact
= initial velocity
= v_i

v_2 = velocity after impact
= final velocity
= v_f

Under these conditions, the *Coefficient of Restitution Equation* reduces to:

$$\begin{aligned} \varepsilon &= -\frac{v_1 - v_2}{V_1 - V_2} \\ &= \frac{v_2 - v_1}{V_1 - V_2} \\ &= \frac{2v_2}{-2V_2} \\ &= -\frac{2v_f}{2v_i} \\ &= -\frac{v_f}{v_i} \end{aligned}$$

Hence, by substituting for V_2, v_2, and ε as indicated above, the *Work Relationship in Terms of Mass, Velocity, and Degree of Elasticity* evolves into the following equation:

$$\begin{aligned} W &= mv_i(v_i - v_f)(1 + \frac{v_f}{v_i}) \\ &= mv_i(v_i - v_f)(\frac{v_i + v_f}{v_i}) \\ &= m(v_i{}^2 - v_f{}^2) \end{aligned}$$

This may be verified by substituting as follows into the *Work Performed Equation* re-listed below, where:

$$\mathrm{V_1 = - V_2 = - v_i}$$
$$\mathrm{v_1 = - v_2 = - v_f}$$
$$\mathrm{m_1 = m_2 = m}$$

$$\begin{aligned} W &= \frac{1}{2}m_1V_1{}^2 + \frac{1}{2}m_2V_2{}^2 - (\frac{1}{2}m_1v_1{}^2 + \frac{1}{2}m_2v_2{}^2) \\ &= \frac{1}{2}m(-v_i)^2 + \frac{1}{2}m(v_i)^2 - [\frac{1}{2}m(-v_f)^2 + \frac{1}{2}m(v_f)^2] \\ &= \frac{1}{2}mv_i{}^2 + \frac{1}{2}mv_i{}^2 - (\frac{1}{2}mv_f{}^2 + \frac{1}{2}mv_f{}^2) \\ &= mv_i{}^2 - mv_f{}^2 \end{aligned}$$

Complex Quadratic Equation for Work Produced when Two Particles of Constant Mass and Velocity Collide.

$$W - m(v_i{}^2 - v_f{}^2) = 0$$

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3.4.3. *Analysis.*

From the *Complex Quadratic Equation for Work Produced when Two Particles of Constant Mass, Velocity and Restitution Collide* above, it is deduced that work, 'W' increases with a decreasing *Coefficient of Restitution*, 'ε'. This indicates that the amount of *work accumulated* is *greater* for collisions which are *more inelastic*.

This is analogous to a situation where a base ball is hit by a bat, and thereafter, by a rubber band. Naturally, the former contact precipitates more noise, heat dissipation, and greater moment requiring resistance by the human body; all indicative of greater work being performed.

Likewise, from *Complex Quadratic Equation for Work Produced when Two Particles of Constant Mass and Velocity Collide* above, a greater amount of work results when the initial velocity before collision, 'v_i' becomes increased.

And lastly, the resulting velocity after the collision, 'v_f' reduces as greater percentages of initial velocities are converted into work.

The *well-known equations* re-stated in *Section 3.4.1*, although appearing in prior literature, now, for the first time may benefit from the added ***Complex Quadratic Equation*** perspective afforded by this treatise.

TRUE SCANS

SECTION 4. *COMPLEX QUADRATIC FUNCTIONS.*

Complex Quadratic Functions are curves that may be represented by *linear* and *quadratic* terms of multiple unknowns.

4.1. *Complex Quadratic Function* Development.

Equation 19 shown below represents the corresponding *Complex Quadratic Function* for the *Complex Quadratic Equation 14*. It may be established simply by replacing the zero appearing on the right-hand side of *Equation 14* by the variable 'y'.

Equation 19. *The Complex Quadratic Function for Equation 14.*

$$z_1^{\,2} z_2 + z_1 z_2^{\,2} - (3\zeta) z_1 z_2 - \zeta = y$$

Under the particular circumstance when $\zeta = \sqrt{3}$, *Equation 19* simplifies to:

Equation 20. *Expression for Equation 19 when ζ = √3.*

$$z_1^{\,2} z_2 + z_1 z_2^{\,2} - 3\sqrt{3} z_1 z_2 - \sqrt{3} = y$$

Once setting $z_2 = 1$ in *Equation 20*, it again reduces to:

Equation 21. *Expression for Equation 20 when $z_2 = 1$.*

$$z_1^{\,2} + (1 - 3\sqrt{3}) z_1 - \sqrt{3} \;=\; y$$

The two z_1 roots for *Equation 21* may be ascertained via *Quadratic Formula*, once setting 'y' = 0.

In short, this *Equation 21 Quadratic Function* represents a reduced *Complex Quadratic Function [Ref. Equation 20]* that may be solved via *Quadratic Formula* once setting y equal to zero.

4.2. Charting.

Table 4 charts *Equation 20* for the *particular condition* when its z_2 value is equal to 1. Accordingly, notice that all z_2 entries equal 1.

Table 5, *Table 6*, and *Table 7*, instead chart *Equation 20* for other *particular conditions* when their respective z_2 values are equal to the following:

$z_2 = \tan 20^{\circ} = 0.363970234$ *[Ref. Table 3]*

$z_2 = \tan 140^{\circ} = -0.839099631$ *[Ref. Table 3]*

$z_2 = \tan 260^{\circ} = 5.67128182$ *[Ref. Table 3]*

Table 4. Equation 20 *Complex Quadratic Function* for $z_2 = 1$.

z_1	z_2	$z_1^2 z_2$	$z_1 z_2^2$	$-3\sqrt{3}z_1z_2$	$-\sqrt{3}$	$z_1^2z_2 + z_1z_2^2 - 3\sqrt{3}z_1z_2 - \sqrt{3} = y$
5.67128182	1	32.16343748	5.67128182	-29.46884477	-1.732050808	6.6338237
5	1	25	5	-25.98076211	-1.732050808	2.2871871
4.574762424	*1*	20.92845124	4.574762424	-23.77116285	-1.732050808	*0*
4	1	16	4	-20.78460969	-1.732050808	-2.51666
3	1	9	3	-15.58845727	-1.732050808	-5.320508
2	1	4	2	-10.39230485	-1.732050808	-6.124356
1	1	1	1	-5.196152423	-1.732050808	-4.928203
0.75	1	0.5625	0.75	-3.897114317	-1.732050808	-4.316665
0.5	1	0.25	0.5	-2.598076211	-1.732050808	-3.580127
0.363970234	1	0.132474331	0.363970234	-1.891244813	-1.732050808	-3.126851
0.25	1	0.0625	0.25	-1.299038106	-1.732050808	-2.718589
0	1	0	0	0	-1.732050808	-1.732051
-0.25	1	0.0625	-0.25	1.299038106	-1.732050808	-0.620513
-0.378610001	*1*	0.143345533	-0.378610001	1.967315274	-1.732050808	*0*
-0.5	1	0.25	-0.5	2.598076211	-1.732050808	0.6160254
-0.75	1	0.5625	-0.75	3.897114317	-1.732050808	1.9775635
-0.839099631	1	0.704088191	-0.839099631	4.360089581	-1.732050808	2.4930273
-1	1	1	-1	5.196152423	-1.732050808	3.4641016

Table 5. Equation 20 *Complex Quadratic Function* for z_2 = 0.363970234.

z_1	z_2	$z_1^2 z_2$	$z_1 z_2^2$	$-3\sqrt{3}z_1z_2$	$-\sqrt{3}$	$z_1^2z_2 + z_1z_2^2 - 3\sqrt{3}z_1z_2 - \sqrt{3} = y$
5.67128182	***0.363970234***	11.70653387	0.751299266	-10.72578233	-1.732050808	*0*
5	0.363970234	9.09925585	0.662371656	-9.456224066	-1.732050808	-1.426647
4.574762424	0.363970234	7.617333294	0.606038593	-8.651995706	-1.732050808	-2.160675
4	0.363970234	5.823523744	0.529897325	-7.564979253	-1.732050808	-2.943609
3	0.363970234	3.275732106	0.397422994	-5.67373444	-1.732050808	-3.73263
2	0.363970234	1.455880936	0.264948662	-3.782489626	-1.732050808	-3.793711
1	0.363970234	0.363970234	0.132474331	-1.891244813	-1.732050808	-3.126851
0.75	0.363970234	0.204733257	0.099355748	-1.41843361	-1.732050808	-2.846395
0.5	0.363970234	0.090992559	0.066237166	-0.945622407	-1.732050808	-2.520443
0.363970234	0.363970234	0.048216713	0.048216713	-0.688356817	-1.732050808	-2.323974
0.25	0.363970234	0.02274814	0.033118583	-0.472811203	-1.732050808	-2.148995
0	0.363970234	0	0	0	-1.732050808	-1.732051
-0.25	0.363970234	0.02274814	-0.033118583	0.472811203	-1.732050808	-1.26961
-0.378610001	0.363970234	0.052173507	-0.050156107	0.716044201	-1.732050808	-1.013989
-0.5	0.363970234	0.090992559	-0.066237166	0.945622407	-1.732050808	-0.761673
-0.75	0.363970234	0.204733257	-0.099355748	1.41843361	-1.732050808	-0.20824
-0.839099631	***0.363970234***	0.256267144	-0.111159162	1.586942825	-1.732050808	*0*
-1	0.363970234	0.363970234	-0.132474331	1.891244813	-1.732050808	0.3906899

Table 6. Equation 20 *Complex Quadratic Function* for z_2 = -0.839099631.

z_1	z_2	$z_1^2 z_2$	$z_1 z_2^2$	$-3\sqrt{3}z_1 z_2$	$-\sqrt{3}$	$z_1^2 z_2 + z_1 z_2^2 - 3\sqrt{3}z_1 z_2 - \sqrt{3} = y$
5.67128182	***-0.839099631***	-26.98832852	3.993082556	24.72729677	-1.732050808	*0*
5	-0.839099631	-20.97749078	3.520440954	21.8004479	-1.732050808	2.6113473
4.574762424	-0.839099631	-17.56105571	3.221036198	19.94637398	-1.732050808	3.8743037
4	-0.839099631	-13.4255941	2.816352763	17.44035832	-1.732050808	5.0990662
3	-0.839099631	-7.551896679	2.112264572	13.08026874	-1.732050808	5.9085858
2	-0.839099631	-3.356398524	1.408176381	8.720179161	-1.732050808	5.0399062
1	-0.839099631	-0.839099631	0.704088191	4.360089581	-1.732050808	2.4930273
0.75	-0.839099631	-0.471993542	0.528066143	3.270067185	-1.732050808	1.594089
0.5	-0.839099631	-0.209774908	0.352044095	2.18004479	-1.732050808	0.5902632
0.363970234	***-0.839099631***	-0.111159162	0.256267144	1.586942825	-1.732050808	*0*
0.25	-0.839099631	-0.052443727	0.176022048	1.090022395	-1.732050808	-0.51845
0	-0.839099631	0	0	0	-1.732050808	-1.732051
-0.25	-0.839099631	-0.052443727	-0.176022048	-1.090022395	-1.732050808	-3.050539
-0.378610001	-0.839099631	-0.120281184	-0.266574831	-1.65077352	-1.732050808	-3.76968
-0.5	-0.839099631	-0.209774908	-0.352044095	-2.18004479	-1.732050808	-4.473915
-0.75	-0.839099631	-0.471993542	-0.528066143	-3.270067185	-1.732050808	-6.002178
-0.839099631	-0.839099631	-0.590800141	-0.590800141	-3.658549558	-1.732050808	-6.572201
-1	-0.839099631	-0.839099631	-0.704088191	-4.360089581	-1.732050808	-7.635328

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Table 7. Equation 20 *Complex Quadratic Function* for z_2 = 5.67128182.

z_1	z_2	$z_1^2z_2$	$z_1z_2^2$	$-3\sqrt{3}z_1z_2$	$-\sqrt{3}$	$z_1^2z_2 + z_1z_2^2 - 3\sqrt{3}z_1z_2 - \sqrt{3} = y$
5.67128182	5.67128182	182.4079183	182.4079183	-167.1261236	-1.732050808	195.95766
5	5.67128182	141.7820455	160.8171874	-147.3442238	-1.732050808	153.52296
4.574762424	5.67128182	118.691145	147.1400852	-134.8129637	-1.732050808	129.28622
4	5.67128182	90.74050912	128.6537499	-117.8753791	-1.732050808	99.786829
3	5.67128182	51.04153638	96.49031245	-88.40653431	-1.732050808	57.393264
2	5.67128182	22.68512728	64.32687496	-58.93768954	-1.732050808	26.342262
1	5.67128182	5.67128182	32.16343748	-29.46884477	-1.732050808	6.6338237
0.75	5.67128182	3.190096024	24.12257811	-22.10163358	-1.732050808	3.4789898
0.5	5.67128182	1.417820455	16.08171874	-14.73442238	-1.732050808	1.033066
0.363970234	***5.67128182***	0.751299266	11.70653387	-10.72578233	-1.732050808	*0*
0.25	5.67128182	0.354455114	8.04085937	-7.367211192	-1.732050808	-0.703948
0	5.67128182	0	0	0	-1.732050808	-1.732051
-0.25	5.67128182	0.354455114	-8.04085937	7.367211192	-1.732050808	-2.051244
-0.378610001	5.67128182	0.812952914	-12.1773991	11.15719935	-1.732050808	-1.939298
-0.5	5.67128182	1.417820455	-16.08171874	14.73442238	-1.732050808	-1.661527
-0.75	5.67128182	3.190096024	-24.12257811	22.10163358	-1.732050808	-0.562899
-0.839099631	***5.67128182***	3.993082556	-26.98832852	24.72729677	-1.732050808	*0*
-1	5.67128182	5.67128182	-32.16343748	29.46884477	-1.732050808	1.2446383

4.3. Graphs.

The *Equation 19 Complex Quadratic Function* now can be plotted. This enables actual viewing of *the very curves* that *Equation 14* roots apply to under the particular circumstance when $\zeta = \sqrt{3}$.

Figures 3 thru 6 depict graphs for *Tables 4 thru 7*, respectively. They map z_1 on their respective x-axes and give corresponding values for $z_1^2 z_2 + z_1 z_2^2 - 3\sqrt{3} z_1 z_2 - \sqrt{3}$ on their y-axis for:

$z_2 = 1$ *[Ref. Figure 3]*
$z_2 = 0.363970234$ *[Ref. Figure 4]*
$z_2 = -0.839099631$ *[Ref. Figure 5]*
$z_2 = 5.67128182$ *[Ref. Figure 6]*

Corresponding roots set values are as follows *(Ref. Table 3)*:

For $z_2 = 1$,
$z_1 = 4.574762424$ and -0.378610001

For $z_2 = 0.363970234$,
$z_1 = -0.839099631$ and 5.67128182

For $z_2 = -0.839099631$,
$z_1 = 0.363970234$ and 5.67128182

For $z_2 = 5.67128182$,
$z_1 = 0.363970234$ and -0.839099631

Each of the four independent root sets noted above arise when their respective curves cross the x-axis – i.e.; when y = 0. All four root sets apply to *Equation 20*, or for that matter, *Equation 19* when $\zeta = \sqrt{3}$.

For each particular z_2 value, a reduced *Complex Quadratic Function* may be calculated for *Equation 20* that may be solved via *Quadratic Formula* once setting y equal to zero. These determinations are denoted as labels on the respective y-axes of *Figures 3 thru 6*, respectively. As <u>reduced</u> *Complex Quadratic Functions*, or *Quadratic Functions*, only two roots are produced at points where the respective curves cross their x-axes.

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Naturally, the ordinate label for *Figure 3* is denoted by *Equation 21*.

For *Figure 4*, ordinate label development is shown below:

$$z_1^{\,2}z_2 + z_1z_2^{\,2} - 3\sqrt{3}z_1z_2 - \sqrt{3} = y \quad \text{[Ref. Equation 20]}$$

$$z_2z_1^{\,2} + (z_2^{\,2} - 3\sqrt{3}z_2)z_1 - \sqrt{3} = y$$

$$(0.363970234)z_1^{\,2} + [(0.363970234)^2 - 3\sqrt{3}(0.363970234)]z_1 - \sqrt{3} = y$$

$$(0.363970234)z_1^{\,2} + [(0.363970234)^2 - 3\sqrt{3}(0.363970234)]z_1 - \sqrt{3} = y$$

$$(0.363970234)z_1^{\,2} - 1.758770483z_1 - \sqrt{3} = y$$

Where, for *Figure* 5:

$$z_1^{\,2}z_2 + z_1z_2^{\,2} - 3\sqrt{3}z_1z_2 - \sqrt{3} = y$$

$$z_2z_1^{\,2} + (z_2^{\,2} - 3\sqrt{3}z_2)z_1 - \sqrt{3} = y$$

$$(-0.839099631)z_1^{\,2} + [(-0.839099631)^2 - 3\sqrt{3}(-0.839099631)]z_1 - \sqrt{3} = y$$

$$(-0.839099631)z_1^{\,2} + 5.064177772z_1 - \sqrt{3} = y$$

And, for *Figure 6*:

$$z_1^{\,2}z_2 + z_1z_2^{\,2} - 3\sqrt{3}z_1z_2 - \sqrt{3} = y \quad \text{[Ref. Equation 20]}$$

$$z_2z_1^{\,2} + (z_2^{\,2} - 3\sqrt{3}z_2)z_1 - \sqrt{3} = y$$

$$(5.67128182)z_1^{\,2} + [(5.67128182)^2 - 3\sqrt{3}(5.67128182)]z_1 - \sqrt{3} = y$$

$$(5.67128182)z_1^{\,2} + 2.694592711z_1 - \sqrt{3} = y$$

Figure 3. Graph of $z_1^2z_2 + z_1z_2^2 - 3\sqrt{3}z_1z_2 - \sqrt{3} = y$, for $z_2 = 1$.

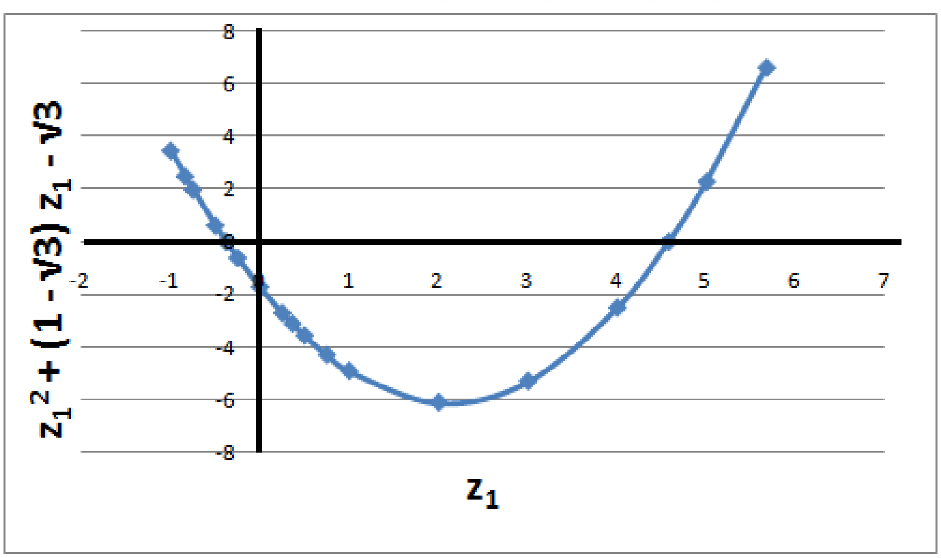


Figure 4. Graph of $z_1^2z_2 + z_1z_2^2 - 3\sqrt{3}z_1z_2 - \sqrt{3} = y$, for $z_2 = 0.363970234$.

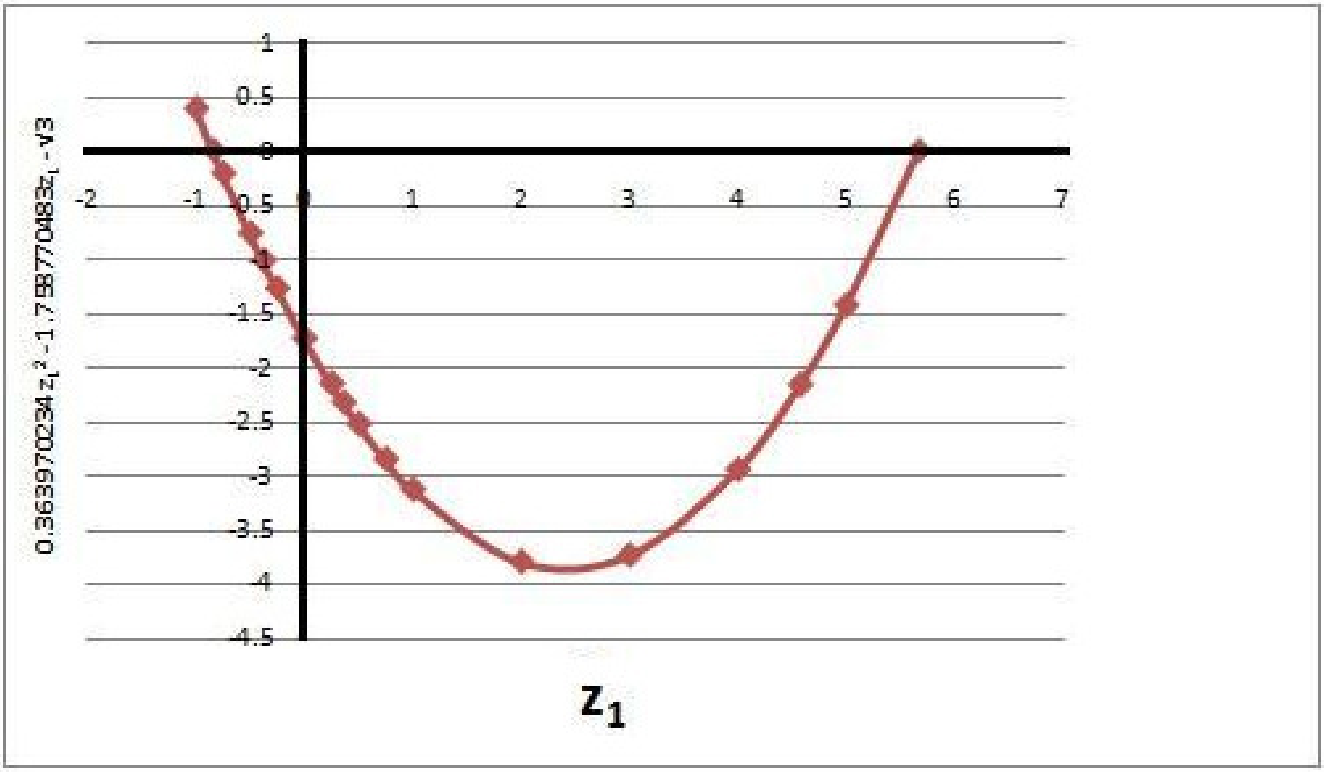


Figure 5. Graph of $z_1^2z_2 + z_1z_2^2 - 3\sqrt{3}z_1z_2 - \sqrt{3} = y$, for $z_2 = -0.839099631$.

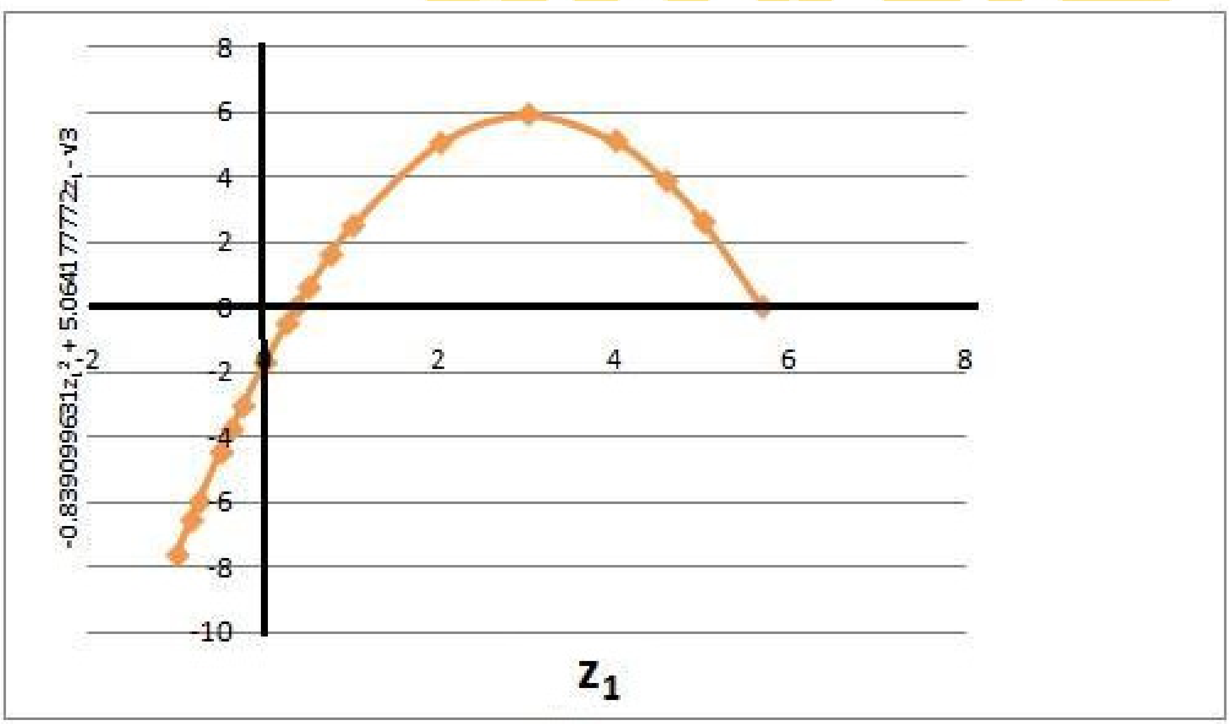
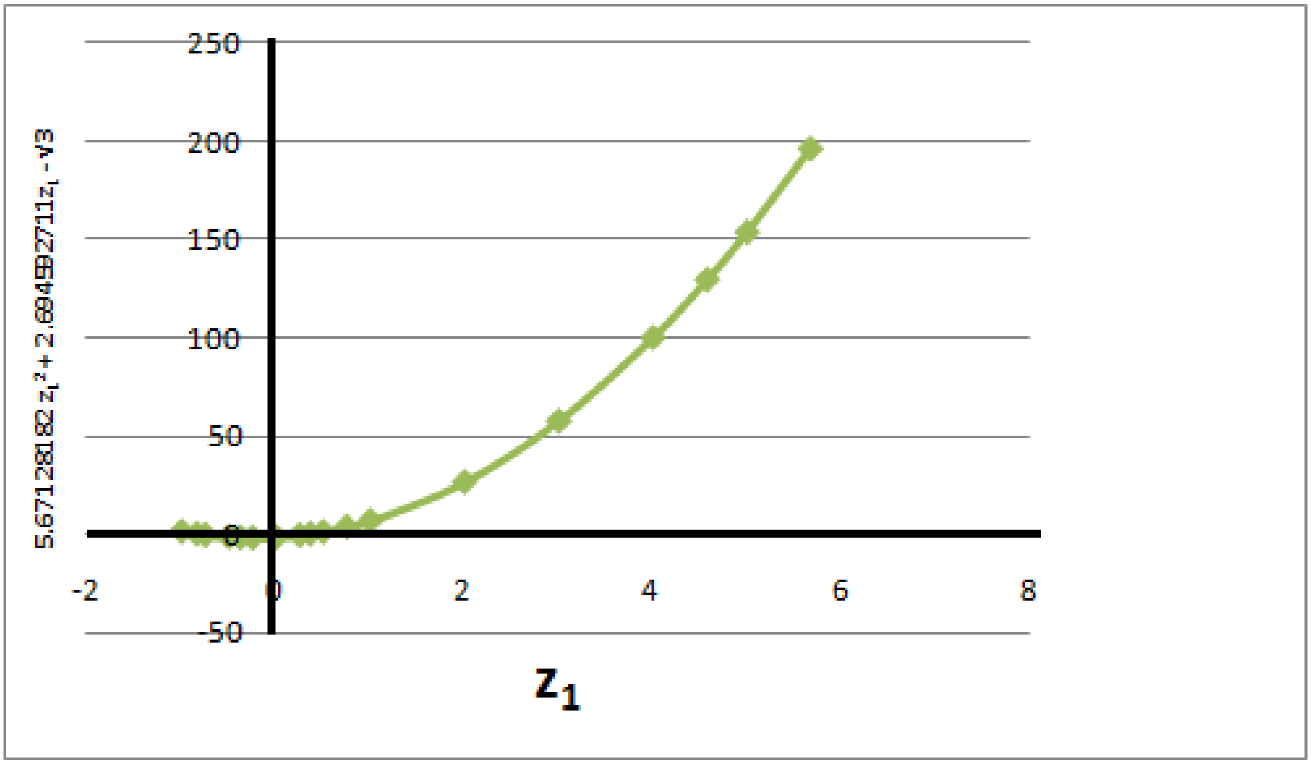


Figure 6. Graph of $z_1^2 z_2 + z_1 z_2^2 - 3\sqrt{3} z_1 z_2 - \sqrt{3} = y$, for $z_2 = 5.67128182$



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SECTION 5. *CUBIC EQUATIONS AND ASSOCIATED FUNCTIONS.*

5.1. Cubic Equation Development.

Per *Section 2.4.3*, the roots for *Equation 3* are:

$$z_1 = \tan\theta$$

$$z_2 = \tan(\theta + 120^o)$$

$$z_3 = \tan(\theta + 240^o)$$

This may be interpreted to mean that three distinct roots of z_1, z_2, and z_3 exist which satisfy *Equation 3* for any arbitrary, given value of ζ = tan (3θ). Accordingly, any of these z_1, z_2, or z_3 root nomenclatures may be substituted back into *Equation 3* for the tan θ in order to produce, or render, the following two sets of equations:

Equation 22. ***Equation 3 Expression when 'z_1' = tangent θ.***

$$z_1^{\,3} = 3(z_1) - \zeta(1 - 3z_1^{\,2})$$

Equation 23. ***Equation 3 Expression for 'z_2' = tan (θ + 120°) or 'z_3' = tan (θ + 240°).***

$$z_2^{\,3} = 3(z_2) - \zeta(1 - 3z_2^{\,2})$$

Or,

$$z_3^{\,3} = 3(z_3) - \zeta(1 - 3z_3^{\,2})$$

Secondly, when considering *Equation 22* it follows that:

$$z_1^{\,3} - 3(z_1) + \zeta(1 - 3z_1^{\,2}) = 0$$

And, when $\zeta = \sqrt{3}$, the following equation is arrived at:

Equation 24. ***Expression for Equation 22 when ζ = √3.***

$$z_1^{\,3} - 3(z_1) + \sqrt{3}(1 - 3z_1^{\,2}) = 0$$

5.2. Elevation of *Complex Quadratic Equations* to *Cubic Equations*.

5.2.1. *Equation Development.*

Complex Quadratic Equations may be transformed into a variety of *cubic formats*. This may be demonstrated by re-arranging *Equation 22* and *Equation 23* into the following set of six new representations:

$z_1 = \dfrac{z_1^{\,3} + \zeta\ (1 - 3z_1^{\,2})}{3}$	$z_2 = \dfrac{z_2^{\,3} + \zeta\ (1 - 3z_2^{\,2})}{3}$	$\zeta = \dfrac{3z_1 - z_1^{\,3}}{1 - 3z_1^{\,2}}$
$z_1^{\,2} = \dfrac{z_1^{\,3} - 3z_1 + \zeta}{3\zeta}$	$z_2^{\,2} = \dfrac{z_2^{\,3} - 3z_2 + \zeta}{3\zeta}$	$\zeta = \dfrac{3z_2 - z_2^{\,3}}{1 - 3z_2^{\,2}}$

From these six equations above, any of the left-hand terms may be substituted into *Complex Quadratic Equation 14*, thereby producing transformations *in cubic form*.

With, six equations being available for substitution, a large variety of transformations can be achieved. Derivations for a few of these transformations are given below:

Where,

$$z_1^{\,2}z_2 + z_1 z_2^{\,2} - (3\zeta)z_1 z_2 - \zeta = 0 \qquad \textit{[Ref. Equation 14]}$$

$$z_1^{\,2}z_2 + z_1 z_2^{\,2} - (3\zeta z_2)[\frac{z_1^{\,3} + \zeta(1-3z_1^{\,2})}{3}] - \zeta = 0$$

$$z_1^{\,2}z_2 + z_1 z_2^{\,2} - (\zeta z_2)[z_1^{\,3} + \zeta(1-3z_1^{\,2})] - \zeta = 0$$

$$z_1^{\,2}z_2 + z_1 z_2^{\,2} - \zeta z_1^{\,3}z_2 - \zeta^2 z_2(1-3z_1^{\,2}) - \zeta = 0$$

$$z_1^{\,2}z_2 + z_1 z_2^{\,2} - \zeta z_1^{\,3}z_2 - \zeta^2 z_2 + 3\zeta^2 z_1^{\,2}z_2 - \zeta = 0$$

$$-\zeta z_1^{\,3}z_2 + z_1^{\,2}z_2(1+3\zeta^2) + z_1(z_2^{\,2}) - \zeta(\zeta z_2 + 1) = 0$$

$$z_1^{\,3}z_2 - z_1^{\,2}z_2\frac{(1+3\zeta^2)}{\zeta} - \frac{z_1(z_2^{\,2})}{\zeta} + (\zeta z_2 + 1) = 0$$

$$z_1^{\,3} - z_1^{\,2}\frac{(1+3\zeta^2)}{\zeta} - \frac{z_1 z_2}{\zeta} + \frac{\zeta z_2 + 1}{z_2} = 0$$

Where,

$$z_1^{\,2}z_2 + z_1 z_2^{\,2} - (3\zeta)z_1 z_2 - \zeta = 0 \qquad \textit{[Ref. Equation 14]}$$

$$z_1^{\,2}z_2 + z_1(\frac{z_2^{\,3} - 3z_2 + \zeta}{3\zeta}) - (3\zeta)z_1 z_2 - \zeta = 0$$

$$3\zeta z_1^{\,2}z_2 + z_1(z_2^{\,3} - 3z_2 + \zeta) - (3\zeta)^2 z_1 z_2 - 3\zeta^2 = 0$$

$$z_2^{\,3} - 3z_2 + \zeta + 3\zeta z_1 z_2 - (3\zeta)^2 z_2 - \frac{3\zeta^2}{z_1} = 0$$

$$z_2^{\,3} + 3(\zeta z_1 - 1 - \zeta^2)z_2 + [\frac{\zeta z_1 - 3\zeta^2}{z_1}] = 0$$

Where,

$$z_1^{\,2}z_2 + z_1 z_2^{\,2} - (3\zeta)z_1 z_2 - \zeta = 0 \qquad \textit{[Ref. Equation 14]}$$

$$z_1^{\,2}z_2 + z_1 z_2^{\,2} - (3)(\frac{3z_2 - z_2^{\,3}}{1-3z_2^{\,2}})z_1 z_2 - \frac{3z_2 - z_2^{\,3}}{1-3z_2^{\,2}} = 0$$

$$(z_1^{\,2}z_2 + z_1 z_2^{\,2})(1-3z_2^{\,2}) - (3)(3z_2 - z_2^{\,3})z_1 z_2 - (3z_2 - z_2^{\,3}) = 0$$

$$(z_1^{\,2}z_2 + z_1 z_2^{\,2} - 3z_1^{\,2}z_2^{\,3} - 3z_1 z_2^{\,4}) - (9z_1 z_2^{\,2} - 3z_1 z_2^{\,4}) - (3z_2 - z_2^{\,3}) = 0$$

$$(z_1^{\,2}z_2 + z_1 z_2^{\,2} - 3z_1^{\,2}z_2^{\,3} - 3z_1 z_2^{\,4}) + (3z_1 z_2^{\,4} - 9z_1 z_2^{\,2}) + (z_2^{\,3} - 3z_2) = 0$$

$$(1-3z_1^{\,2})z_2^{\,3} - 8z_1 z_2^{\,2} + (z_1^{\,2} - 3)z_2 = 0$$

Which further reduces to:

$$(1-3z_1^{\ 2})z_2^{\ 2}-8z_1z_2+(z_1^{\ 2}-3)=0$$

$$z_2^{\ 2}-(\frac{8z_1}{1-3z_1^{\ 2}})z_2+\frac{z_1^{\ 2}-3}{1-3z_1^{\ 2}}=0$$

Where,

$$z_1^{\ 2}z_2+z_1z_2^{\ 2}-(3\zeta)z_1z_2-\zeta=0 \qquad \textit{[Ref. Equation 14]}$$

$$z_1^{\ 2}z_2+z_1z_2^{\ 2}-(3)[\frac{3z_2-z_2^{\ 3}}{1-3z_2^{\ 2}}]z_1z_2-\frac{3z_1-z_1^{\ 3}}{1-3z_1^{\ 2}}=0$$

$$(z_1^{\ 2}z_2+z_1z_2^{\ 2})(1-3z_2^{\ 2})-(3)[3z_2-z_2^{\ 3}]z_1z_2-(3z_1-z_1^{\ 3})=0$$

$$(z_1^{\ 2}z_2+z_1z_2^{\ 2}-3z_1^{\ 2}z_2^{\ 3}-3z_1z_2^{\ 4})-(9z_1z_2^{\ 2}-3z_1z_2^{\ 4})-(3z_1-z_1^{\ 3})=0$$

$$(z_1^{\ 2}z_2+z_1z_2^{\ 2}-3z_1^{\ 2}z_2^{\ 3}-3z_1z_2^{\ 4})+(3z_1z_2^{\ 4}-9z_1z_2^{\ 2})+(z_1^{\ 3}-3z_1)=0$$

$$z_1^{\ 3}+(z_2-3z_2^{\ 3})z_1^{\ 2}-(8z_2^{\ 2}+3)z_1=0$$

Which reduces to:

$$z_1^{\ 2}+(z_2-3z_2^{\ 3})z_1-(8z_2^{\ 2}+3)=0$$

5.2.2. *Verification.*

From the above section, it is obvious that *Complex Quadratic Equation* $(1-3z_1^{\ 2})z_2^{\ 2}-8z_1z_2+(z_1^{\ 2}-3)=0$ represents a reduction of the *Cubic Equation* $(1-3z_1^{\ 2})z_2^{\ 3}-8z_1z_2^{\ 2}+(z_1^{\ 2}-3)z_2=0$ which resulted as a *transformation* of the original *Equation 14 Complex Quadratic* $z_1^{\ 2}z_2+z_1z_2^{\ 2}-(3\zeta)z_1z_2-\zeta=0$.

Nevertheless, it should not be attempted to resolve either of such transformations in terms of their *Equation 14* predecessor simply because they represent ***distinct sets, or families of constituent roots.***

Hence, unless more specific information becomes afforded pertaining to which particular set of unknowns requires evaluation, then these *Complex Quadratic Equations*, as **stand-alone documents**, even in combination, cannot render further detailed resolution.

However, once additional information becomes supplied, these equations prove useful.

For example for

$$\mathrm{z}_1=\cos 20^{\circ}=0.363970234:$$

$$z_2^{\ 2}-[\frac{8z_1}{(1-3z_1^{\ 2})}]z_2+\frac{(z_1^{\ 2}-3)}{(1-3z_1^{\ 2})}=0$$

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$$z_2^{\,2} - [\frac{8(0.363970234)}{1-3(0.363970234)^2}]z_2 + \frac{(0.363970234)^2 - 3}{1-3(0.363970234)^2} = 0$$

$$z_2^{\,2} - \frac{1.758770484}{0.363970234}z_2 - \frac{1.732050808}{0.363970234} = 0$$

$$z_2^{\,2} - 4.83218219z_2 - 4.758770483 = 0$$

Or, from the *Quadratic Formula*:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z_2 = \frac{+4.83218219 \pm \sqrt{(4.83218219)^2 - 4(1)(-4.758770483)}}{2(1)}$$

$$= \frac{+4.83218219 \pm \sqrt{23.34998472 + 19.03508193}}{2(1)}$$

$$= \frac{+4.83218219 \pm \sqrt{42.38506665}}{2(1)}$$

$$= +5.67128182, -0.839099631$$

$$= \tan(4\theta), -\tan(2\theta)$$

This result is identical to the z_2 root determination afforded in *Table 2*. Hence, from the above analysis, this *transformed Complex Quadratic Equation* also is validated.

5.3. Cubic Function Development.

Replacing the zero appearing on the right-hand side of *Cubic Equation 24* by the variable 'y' establishes its corresponding *Cubic Function* as follows:

Equation 25. *The Cubic Function for Equation 24.*

$$z_1^{\,3} - 3(z_1) + \sqrt{3}(1 - 3z_1^{\,2}) = y$$

5.4 Charting.

Table 8 applies five distinct values of z_1 to *Equation 24*. The first two z_1 entries represent values of the roots to *Equation 18*; the last three represent values for tan 20°, tan (20° + 120°), and tan (20° + 240°), respectively.

Table 8. *Equation 24* Listing for Five Independent z_1 Values

z_1	z_1^3	$-3z_1$	$\sqrt{3}(1-3z_1^2)$	$z_1^3 - 3z_1 + \sqrt{3}(1-3z_1^2) = 0$
4.574762424	95.74269231	-13.72428727	-107.0153718	-24.99696678
-0.378610001	-0.054272052	1.135830003	0.98720557	2.068763521
0.363970234	0.048216713	-1.091910702	1.04369399	0
-0.839099631	-0.590800141	2.517298893	-1.926498751	0
5.67128182	182.4079183	-17.01384546	-165.3940728	0

Columns two, three, and four present calculated values for each of the terms expressed in *Equation 24*. The last column gives respective summations for the entire equation.

Since roots occur only when selected z_1 values satisfy *Equation 24*, they become realized when last column summations equal zero. As indicated, only the bottom three rows reflect this. Hence, the top two z_1 selections do not represent roots for *Equation 24*.

And, since *Equation 24* qualifies as a *Cubic Equation*, naturally it exhibits only three roots.

Moreover, *Table 9* depicts calculations of the *Equation 25 Cubic Function* for same z_1 entries applied in *Tables 4 thru 7*, respectively.

Columns two thru four depict calculated values for respective *Equation 25* terms. Again, the last column in the table enumerates totals for the *Equation 25* y function.

Table 9. Plot of *Equation 25 Cubic Function.*

z_1	z_1^3	$-3z_1$	$\sqrt{3}(1-3z_1^2)$	$z_1^3 - 3z_1 + \sqrt{3}(1-3z_1^2) = y$
5.67128182	182.4079183	-17.01384546	-165.3940728	0
5	125	-15	-128.1717598	-18.17175979
4.574762424	95.74269231	-13.72428727	-107.0153718	-24.99696678
4	64	-12	-81.40638798	-29.40638798
3	27	-9	-45.03332101	-27.03332101
2	8	-6	-19.05255889	-17.05255889
1	1	-3	-3.464101616	-5.464101616
0.75	0.421875	-2.25	-1.190784931	-3.018909931
0.5	0.125	-1.5	0.433012702	-0.941987298
0.363970234	0.048216713	-1.091910702	1.043693991	0
0.25	0.015625	-0.75	1.407291282	0.672916282
0	0	0	1.732050808	1.732050808
-0.25	-0.015625	0.75	1.407291282	2.141666282
-0.378610001	-0.054272052	1.135830003	0.98720557	2.068763521
-0.5	-0.125	1.5	0.433012702	1.808012702
-0.75	-0.421875	2.25	-1.190784931	0.63734007
-0.839099631	-0.590800141	2.517298893	-1.926498751	0
-1	-1	3	-3.464101616	-1.464101616

Per *Table 8*, the three z_1 roots for the *Equation 24* cubic are identified as 0.363970234, -0.839099631, and 5.67128182 respectively.

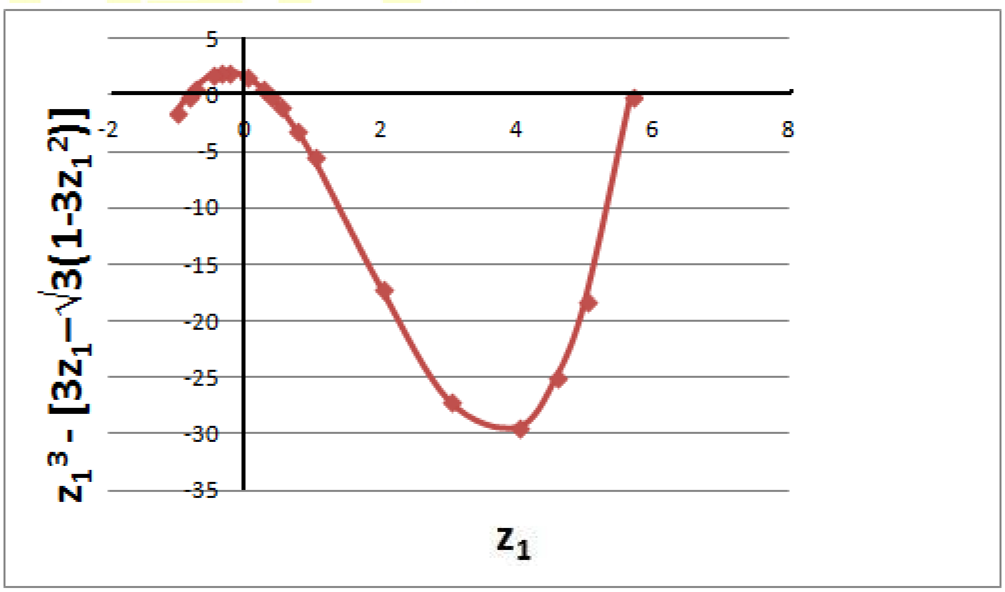
Accordingly, they are depicted as bold, italicized numbers in *Table 9* in order to distinguish them as *Equation 25* roots also.

5.5 Graphs.

Figure 7 presents the plot for *Table 9*. It maps z_1 values on its x-axis and corresponding calculations for $z_1^3 - 3(z_1) + \sqrt{3}(1-3z_1^2)$ on its y-axis. As a *Cubic Function,* three roots are illustrated at points where the curve crosses the x-axis (i.e.; where y = 0).

As shown, respective roots when y = 0 are: z_1 = 0.363970234, -0.839099631, and 5.67128182.

Figure 7. Graph of $z_1^3 - 3z_1 + \sqrt{3}(1 - 3z_1^2) = y$.



SECTION 6. *COMPARISON OF COMPLEX QUADRATICS AND CUBICS.*

This section identifies similarities and differences that stem between *Complex Quadratics* and their corresponding *Cubic Transformations.*

In order to assist in this analysis, a Hierarchy Chart has been created which exhibits the following attributes:

- It *categorizes equations* and *functions* by section, where
  - Section 2 depicts *Fundamental Information*
  - Section 3 depicts *Complex Quadratic Equations*
  - Section 4 depicts *Complex Quadratic Functions*
  - Section 5 depicts *Cubic Equations* and *Functions*
- It expresses *parent lineage*, or *paths of development*, which, by quick glance, help to determine various similarities and differences that exist between the equation types expressed above
- It identifies *distinguishing details* that exist between respective equations, in order to rapidly segregate those which possess identical z_1 or ζ values in common.

Equations 1,2, and 4 thru 10 do not appear in *Table 10*. These may be used to contribute to the development of additional, future *Hierarchy Charts* which presently fall outside of the confines of this analysis.

Development of *the various equations and functions appearing* in *Table 10* is summarized below by section:

FOR SECTION 2. FUNDAMENTAL INFORMATION

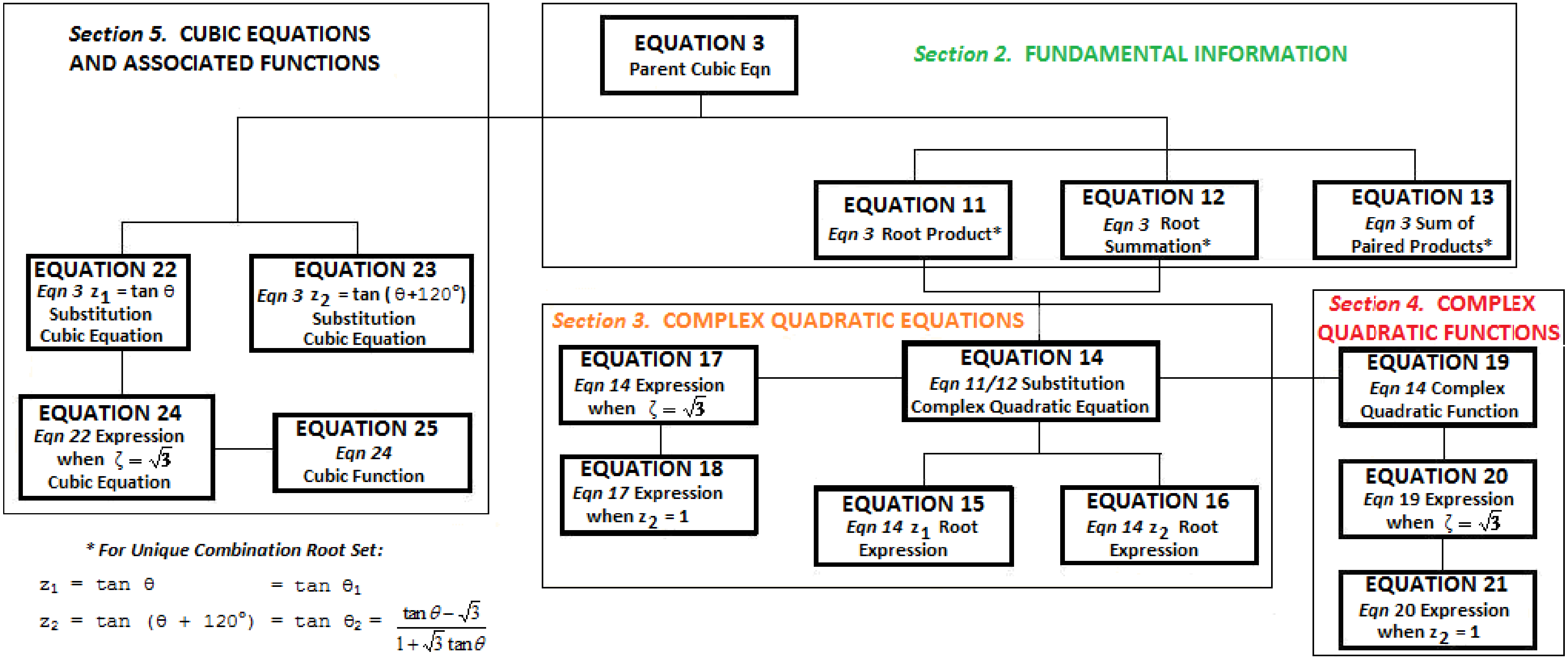
- *Equations 11 thru 13* represent *reductions* of *Cubic Equation 3* based on the fact that its roots can be characterized as follows:

$$z_1 = \tan\theta = \tan\theta_1$$

$$z_2 = \tan(\theta + 120^\circ) = \tan\theta_2 = \frac{\tan\theta - \sqrt{3}}{1+\sqrt{3}\tan\theta}$$

$$z_3 = \tan(\theta + 240^\circ) = \tan\theta_3 = \frac{\tan\theta + \sqrt{3}}{1-\sqrt{3}\tan\theta}$$

Table 10. Hierarchy Chart.



** For Unique Combination Root Set:*

$$z_1 = \tan\theta = \tan\theta_1$$

$$z_2 = \tan(\theta + 120^\circ) = \tan\theta_2 = \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta}$$

$$z_3 = \tan(\theta + 240^\circ) = \tan\theta_3 = \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$$

FOR SECTION 3. *COMPLEX QUADRATIC EQUATIONS.*

- The *Complex Quadratic Equation 14* is merely a *reformatting* of the *Equation 11 and 12* reductions
- *Equations 15 and 16* were derived from *Equation 14 in order to* express its root characterizations in a format representative of the *Quadratic Formula*
- *Equation 17* relates *Equation 14* for the unique case when $\sqrt{3}$ is substituted for ζ
- Likewise, *Equation 18* depicts the *Equation 17* characterization under the particular condition when $z_2 = 1$; or, of course when *Equation 14* exhibits the following values of $\zeta = \sqrt{3}$, and $z_2 = 1$

FOR SECTION 4. *COMPLEX QUADRATIC FUNCTIONS.*

- *Equation 19* represents the *Quadratic Function* for the Complex Quadratic *Equation 14*. It consists of replacing a variable designation of 'y' for the zero on the right-hand side of *Equation 14*
- *Equation 20* depicts the *Equation 19 Complex Quadratic Function* when $\sqrt{3}$ is substituted for ζ. Accordingly, it represents the *Complex Quadratic Function* for *Complex Quadratic Equation 14* when $\zeta = \sqrt{3}$
- Likewise, *Equation 21* represents the *Equation 20 Complex Quadratic Function* when $z_2 = 1$. Accordingly, it represents the *Complex Quadratic Function* for *Complex Quadratic Equation 14* when $\zeta = \sqrt{3}$, and $z_2 = 1$

FOR SECTION 5. *CUBIC EQUATIONS AND ASSOCIATED FUNCTIONS.*

- Substitution of the z_1 thru z_2 roots (repeated below) back into *Equation 3* enables establishment of *Cubic Equations 22 and 23*, respectively.

$$z_1 = \tan\theta = \tan\theta_1$$

$$z_2 = \tan(\theta + 120^\circ) = \tan\theta_2 = \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta}$$

- *Cubic Equation 24* depicts *Equation 22* for the unique case when $\sqrt{3}$ is substituted for ζ.
- *Equation 25* represents the *Cubic Function* for the *Cubic Equation 24*. It consists of replacing a variable designation of 'y' for the zero on the right-hand side of *Equation 24*.

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6.1. Similarities

Independent equations which are derived or stem from the same *parent equation* exhibit *certain similarities* in common. This applies to a whole gamut of derivations and transformations that may be categorized into groups of *Complex Quadratic* and *Cubic Equations*, in addition to their respective *Functions*.

Section 3 Complex Quadratic Equations which *depict* the sum of their terms as zero signify respective equalities for resolving, or solving for, *sets of roots*, or *root families*.

Root sets, as represented *either* in z_1 or z_2, consist of two values each. For z_1 they occur for any constant value of z_2; and vice versa. *Equation 15* and *Equation 16*, the only equations presented in *Section* 3 which are not portrayed to equal zero, calculate respective root sets for any given value of ζ and the other variable via the *Quadratic Formula*.

Complex Quadratic Functions may be converted to their corresponding *Complex Quadratic Equations* by simply setting their respective variable 'y' back to zero. Hence, their *root sets* must represent numerical values which satisfy their corresponding *Complex Quadratic Equations*.

In other words, *root sets* for *Complex Quadratic Functions* and their corresponding *Complex Quadratic Equations* must be exactly the same!

So, *Table 4* focuses upon the unique circumstance when z_2 = 1. *Table 4* roots constitute conditions when z_1 values are such that the *Equation 20 Complex Quadratic Function* totals to, or equals zero.

This is equivalent to solving the *Complex Quadratic Equation 14* for $\zeta = \sqrt{3}$ and $z_2 = 1$; or merely solving the *Complex Quadratic Equation 17* for $z_2 = 1$ *(Ref. Table 3)*; or, lastly solving the *Complex Quadratic Equation 18 [Ref. Table 10]*.

Accordingly, *Table 3*, and *Table 4*, respectively, share a common root set. This is to be expected because:

- *Table 3*, becomes synonymous with *a portion* of *Table 4* only for rows when its z_2 value equals unity.
- *Table 4*, becomes synonymous with *a portion* of *Table 3* only for rows when its right-hand column equals zero.

More specifically, this common *root set* occurs when z_1 equals either 4.574762424, or −0.378610001. Naturally, this *Quadratic Equation 18 root set* relates to a corresponding *Equation 21 Quadratic Function [Ref. Table 10]* which may be viewed pictorially via *Figure 3*.

Moreover, according to *Table 10*, *Equation 20* represents the *Complex Quadratic Function* for its corresponding *Complex Quadratic Equation 17*.

Table 3, which appears in *Section 3*, charts four root sets of z_2 for the *Complex Quadratic Equation 17*.

Table 4, *Table 5*, *Table 6*, and *Table 7*, which appear in *Section 4*, identify these same exact four root sets of z_2, respectively, for *Equation 20*.

These common root sets are specified as follows:

Considering the case when ***z_2 is set equal to 1***, the following should hold true:

- Per *Table 3*, respective z_1 entries of 4.574762424 and −0.378610001 depict roots for the *Equation 17* -- <u>*Complex Quadratic Equation*</u>
- Per *Table 4*, respective z_1 entries of 4.574762424 and −0.378610001 depict roots for the *Equation 20* -- <u>*Complex Quadratic Function*</u>

Accordingly, *Table 4* depicts the dual roots z_1 = 4.574762424 and −0.378610001, respectively, as **bold**, *italicized numbers* in order to further distinguish them as *Equation 17* roots.

Considering the case when ***z_2 is set equal to 0.363970234***, the following should hold true:

- Per *Table 3*, respective z_1 entries of −0.839099631 and 5.67128182 depict roots for the *Equation 17* -- <u>*Complex Quadratic Equation*</u>
- Per *Table 5*, respective z_1 entries of −0.839099631 and 5.67128182 depict roots for the *Equation 20* -- <u>*Complex Quadratic Function*</u>

So, *Table 5* depicts the dual roots z_1 = −0.839099631 and 5.67128182, respectively, as **bold**, *italicized numbers* in order to further distinguish them as *Equation 17* roots.

Considering the case when ***z_2* is set equal to -0.839099631**, the following should hold true:

- Per *Table 3*, respective z_1 entries of 0.363970234 and 5.67128182 depict roots for the *Equation 17* -- *Complex Quadratic Equation*
- Per *Table 6*, respective z_1 entries of 0.363970234 and 5.67128182 depict roots for the *Equation 20* -- *Complex Quadratic Function*

So, *Table 6* depicts the dual roots z_1 = 0.363970234 and 5.67128182, respectively, as **bold**, *italicized* numbers in order to further distinguish them as *Equation 17* roots.

Considering the case when ***z_2* is set equal to 5.67128182**, the following should hold true:

- Per *Table 3*, respective z_1 entries of 0.363970234 and -0.839099631 depict roots for the *Equation 17* -- *Complex Quadratic Equation*
- Per *Table 7*, respective z_1 entries of 0.363970234 and -0.839099631 depict roots for the *Equation 20* -- *Complex Quadratic Function*

So, *Table 7* depicts the dual roots z_1 = 0.363970234 and -0.839099631, respectively, as **bold**, *italicized* numbers in order to further distinguish them as *Equation 17* roots.

Table 9 roots occur when z_1 values permit the *Equation 25 Complex Quadratic Function* to equal zero. As such, these roots also must satisfy its corresponding *Complex Quadratic Equation 24*, as depicted *in Table 8*.

Since these *Tables* 8 and 9 appear in *Section 5*, which addresses *Cubic Equations* and their *associated Functions*, they both express the exact same *three distinct roots*. Moreover, these *same roots* also apply to *Tables 5 thru 7*.

Table 5 thru 7 and 9 plots are illustrated in *Figures 4 thru 7*, respectively. Below, they are superimposed in *Figure 8* in order to disclose their *common root set characterizations*.

Respective common roots are given below:

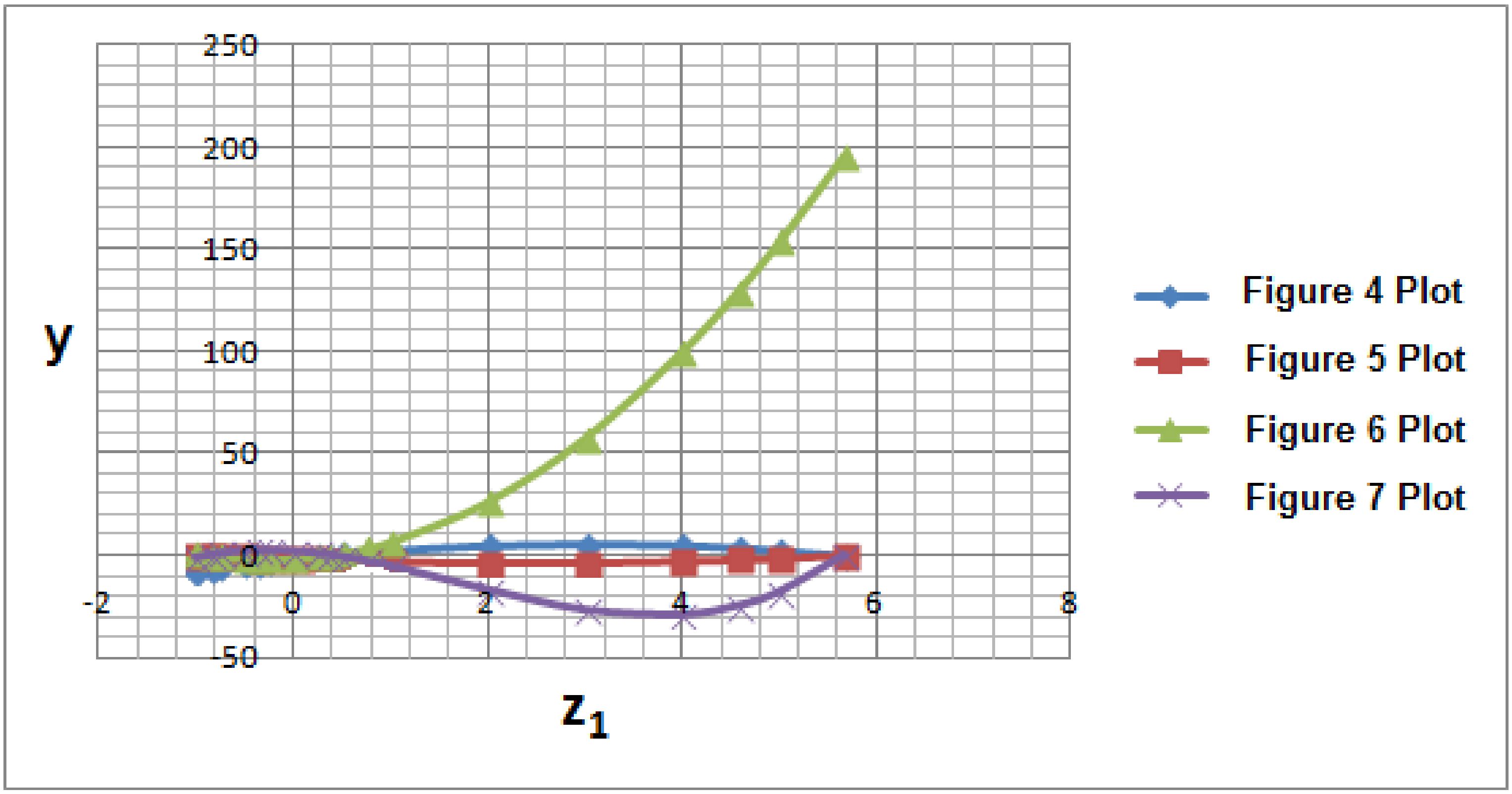
- *Figure 4* roots are -0.839099631 and 5.67128182
- *Figure* 5 roots are 0.363970234 and 5.67128182
- *Figure 6* roots are 0.363970234 and -0.839099631

- *Figure 7* roots are 0.363970234, -0.839099631 and 5.67128182

For the curves described in *Figure* 4, *Figure* 5, and *Figure 6*, y is represented by the *Equation 20 Complex Quadratic Function*; whereby, for *Figure 7*, y is designated by the *Equation 25 Cubic Function*.

Such *Complex Quadratic Functions* referred to in *Figure* 4, *Figure* 5, and *Figure 6* exhibit *only* two roots each. Notice that together they encompass all of the possibilities for identifying two out of three roots of the *Figure 7 Cubic Function* plot.

Figure 8. Common Root Set Relationships.



6.2. Differences.

Since *Section 3 and 4* entries in *Table 10* represent *Quadratic* characterizations while *Equation 3*, as well as, *Section 5* equations and their associated function(s) are *Cubic* in nature, it seems very plausible that certain differences should exist between such representations.

One difference is that *Complex Quadratic Equations* can encompass roots that are not contained in corresponding *Cubic Equations*. For example, *Table 3* demonstrates that the *Complex Quadratic Equation 17* embellishes root sets which are supplemental to those which satisfy *Cubic Equation 24*, as expressed in *Table 8.* Even though *Table 3* lists just four values of z_2, there are an infinite number of other values whose roots sets will not satisfy *Equation 24*, particularly because the latter *Cubic Equation* can contain only three root values.

This is further demonstrated in *Table 9* where roots become realized only when z_1 values permit the *Equation 25* function to equal zero. Per this table, the root set of z_1 equals either 4.574762424, or -0.378610001 does not allow the summation of the *Equation 25* function to equal zero. Hence, although this root set satisfies the *Complex Quadratic Equation 17 (Table 3), it* applies neither to the *Equation 25 Cubic Function*, nor to the *Equation 24 Cubic Equation* that it modifies. Nor does this root set represent roots for the completely independent *Parent Cubic Equation 3* from which *Complex Quadratic Equation 17* originates *[Ref. Table 10]*.

TRUE SCANS

SECTION 7. *LINEARIZING THE CUBIC.*

This section demonstrates that *Cubic Equations* can be expressed not only in terms of *Quadratic* and *Complex Quadratic transformations*, but also as direct *Linear reductions*.

This process may be viewed as actually *skipping over* quadratic representations entirely, or transforming from *Cubic Equation* directly into an associated Linear reduction!

This is accomplished as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\begin{aligned}\sin(2\theta) &= \sin(3\theta - \theta)\\ &= \sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta\\ &= \eta\cos\theta - \tau\sin\theta\end{aligned}$$

Or,

$$2\sin\theta\cos\theta = \eta\cos\theta - \tau\sin\theta$$

$$2 = \frac{\eta}{\sin\theta} - \frac{\tau}{\cos\theta}$$

Then, a *straight line, linear transformation* for *Cubic Equation* entities τ, η, sin θ, and cos θ becomes,

Equation 26. ***Linearization of the Cubic.***

$$\frac{1}{\sin\theta} = \frac{\tau}{\eta}\left(\frac{1}{\cos\theta}\right) + \frac{2}{\eta}$$

Which, of course, assumes its linear form as:

$$y = mx + b$$

Hence, the *Linear Equation 26* characterizes $1/\sin\theta$ and $1/\cos\theta$ in terms of the known coefficients 2, η, and τ, where:

- $\frac{\tau}{\eta} = \frac{1}{\zeta} = \frac{1}{\tan(3\theta)}$ represents the slope of straight line
- $\frac{2}{\eta}$ determines the 'y' intercept
- $\frac{1}{\cos\theta}$ signifies the 'x' coordinate
- $\frac{1}{\sin\theta}$ identifies the corresponding 'y' ordinate

TRUE SCANS

Similarly, *Straight Line Linearization* for y = 1/sin(2θ), and 1/sin (4θ), respectively, for the condition when θ = 20° is established below

Table 11. Linearizing the Cubic Development.

$$\sin(2\theta) = \sin(3\theta - \theta)$$
$$= \sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta$$
$$= \eta\cos\theta - \tau\sin\theta$$

Where:

$\sin(2\theta) = 2\sin\theta\cos\theta$,

Or,

$$\eta\cos\theta - \tau\sin\theta = 2\sin\theta\cos\theta$$
$$\frac{\eta}{\sin\theta} - \frac{\tau}{\cos\theta} = 2$$
$$\frac{1}{\sin\theta} = \frac{\tau}{\eta}\left(\frac{1}{\cos\theta}\right) + \frac{2}{\eta}$$

KNOWN STRAIGHT LINE AC EXTENDED (Ref. Figure 9)

$$\sin(4\theta) = \sin 5\theta = \sin(3\theta + 2\theta)$$
$$= \sin(3\theta)\cos(2\theta) + \cos(3\theta)\sin(2\theta)$$
$$= \eta\cos(2\theta) + \tau\sin(2\theta)$$

Where:

$\sin(4\theta) = 2\sin(2\theta)\cos(2\theta)$,

where:

Or,

$$\eta\cos(2\theta) + \tau\sin(2\theta) = 2\sin(2\theta)\cos(2\theta)$$
$$\frac{\eta}{\sin(2\theta)} + \frac{\tau}{\cos(2\theta)} = 2$$
$$\frac{1}{\sin(2\theta)} = -\frac{\tau}{\eta}\left[\frac{1}{\cos(2\theta)}\right] + \frac{2}{\eta}$$

KNOWN STRAIGHT LINE AD EXTENDED (Ref. Figure 9)

$$\sin(8\theta) = \sin(3\theta + 5\theta)$$
$$= \sin(3\theta)\cos(5\theta) + \cos(3\theta)\sin(5\theta)$$
$$= \sin(3\theta)\cos(100^o) + \cos(3\theta)\sin(100^o)$$
$$= -\sin(3\theta)\cos(80^o) + \cos(3\theta)\sin(80^o)$$
$$= -\sin(3\theta)\cos(4\theta) + \cos(3\theta)\sin(4\theta)$$
$$= -\eta\cos(4\theta) + \tau\sin(4\theta)$$

Where:

$\sin(8\theta) = 2\sin(4\theta)\cos(4\theta)$,

where:

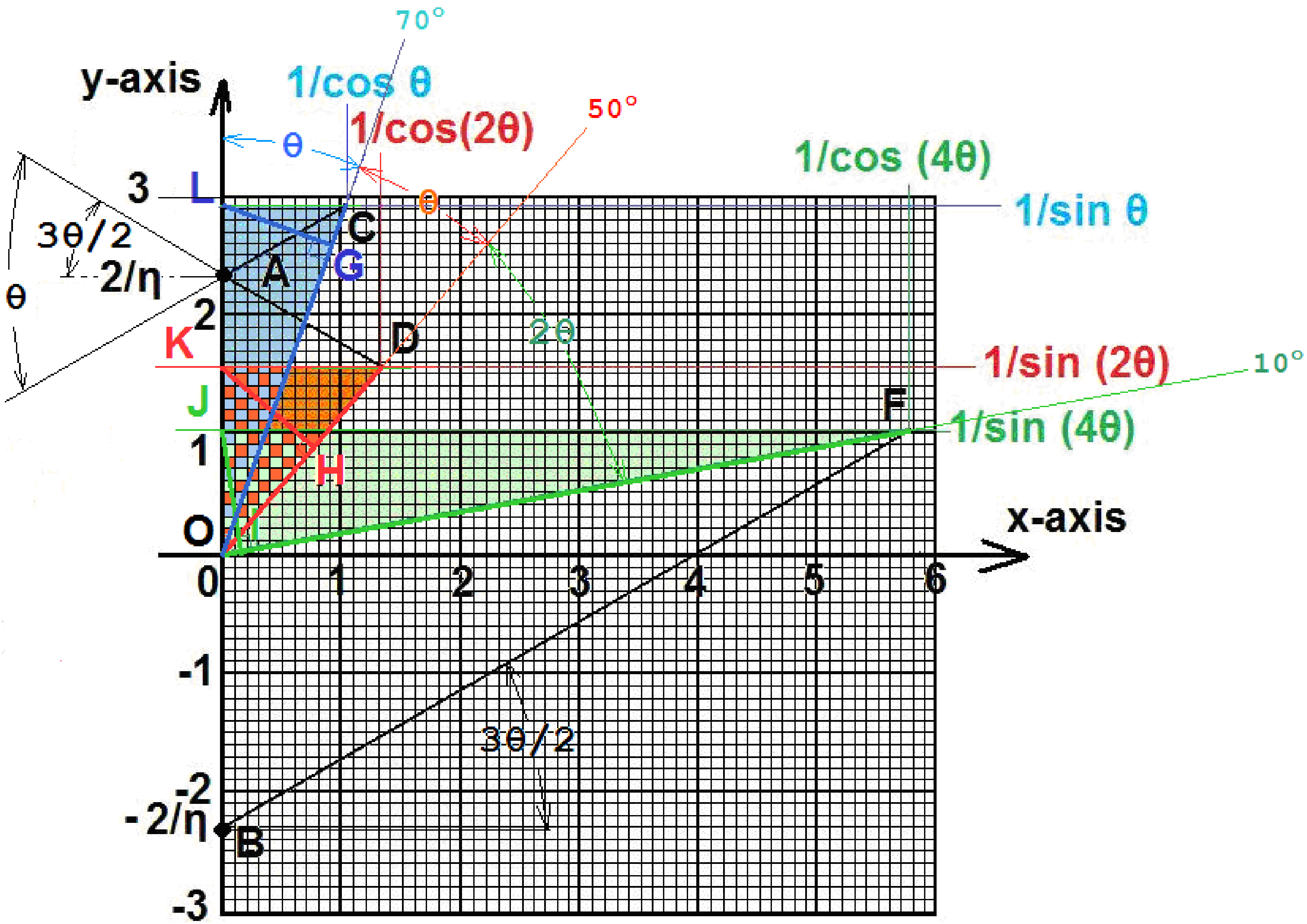
Or,

$$-\eta\cos(4\theta) + \tau\sin(4\theta) = 2\sin(4\theta)\cos(4\theta)$$
$$-\frac{\eta}{\sin(4\theta)} + \frac{\tau}{\cos(4\theta)} = 2$$
$$\frac{1}{\sin(4\theta)} = \frac{\tau}{\eta}\left[\frac{1}{\cos(4\theta)}\right] - \frac{2}{\eta}$$

KNOWN STRAIGHT LINE BF EXTENDED (Ref. Figure 9)

TRUE SCANS

Figure 9. Linearizing the Cubic.



Considering the respective lengths of 1/sinθ, 1/sin(2θ), and 1/sin(4θ) given in *Figure 9*, from the geometry afforded:

LG = 1
KH = 1
JI = 1.

Where,

- LG is perpendicular to line $\overline{OC}$
- KH is perpendicular to line $\overline{OD}$
- JI is perpendicular to line $\overline{OF}$

TRUE SCANS

SECTION 8. *IDENTITIES*.

Identities encompass *indeterminate equations* whose formats defy mathematical resolution.

Such definition applies *even* to *Cubic Equation formats* which express only *singular unknown quantities* such as those enumerated in *Table 12*.

Table 12. Cubic Equivalency Table.

$\cos^3\theta = 3/4\cos\theta + \tau/4$	*[Ref. Equation 1]*
$\sin^3\theta = 3/4\sin\theta - \eta/4$	*[Ref. Equation 2]*
$\tan^3\theta = 3\tan - \zeta(1 - 3\tan^2\theta)$	*[Ref. Equation 3]*

Each of these above equations, in itself, is considered to be extraordinary in that it manifests *only* a *singular unknown* but, nevertheless, still defies mathematical resolution!

Reductions of *Cubic* or even *Higher Order Equations* can be achieved by substituting respective right-hand *lower order terms* of equations *presented in Table 12* for left-hand *cubic equivalencies* appearing in other equations.

For example, with regard to *Quartic Equations*, applicable *cubic expressions expressed in Table 12* need to be substituted for twice, in order to reduce into *Quadratic Equation* format.

In some identities, *the coefficients of all included terms each equate to zero (Ref. Section 8.3)*.

In others, *numerical summations of respective terms* on each side of the equation may equate. *Equality* is still maintained because left-hand side and right-hand terms sum to zero *(Ref. Section 8.4)*.

Hence, such *identities* cannot provide quantitative indication of unknown numerical value. However, they can *validate* that mathematical calculations conducted during the *reduction process* were performed *correctly*!

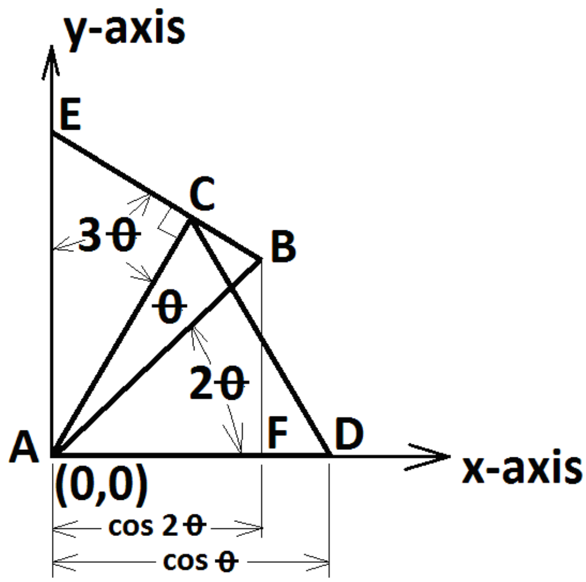
A *constituent geometry* for generating cosine related *identities* in presented in Section 8.1.

The practice of *Mathematical Closure* may be applied in order to resolve such annoying *identities (Ref. Section 8.5)*.

8.1. The Identity Geometry.

Figure 10 portrays a *constituent geometry* for producing, or generating, cosine related *identities*.

Figure 10. Geometry for Generating *Identities*.



Its *construction* centers upon *equilateral triangle ACD* where member $\overline{AD}$ is placed onto the x-axis of a mutually orthogonal coordinate system, with vertex A located at its origin.

Next, *right triangle ACB* is constructed such that its hypotenuse $\overline{AB}$ is equal to unity; also, $\angle BAC$ is to be designated as θ.

Since *equilateral triangle ACD* contains 60° vertices:

$$\angle CAD = 60^\circ$$

$$\frac{\angle CAD}{3} = \frac{60^\circ}{3}$$

$$= 20^\circ$$

Hence, when:

$$\angle BAC = \theta = 20^o$$

Hypotenuse $\overline{AB}$ *trisects* $\angle CAD$ such that:

$$\frac{\angle CAD}{3} = \theta$$

$$\angle CAD = 3\theta$$

Where $\overline{CE}$ represents an extension to line $\overline{BC}$ such that *Point E* represents the straight line’s intersection with the y-axis:

TRANS SCANS

$\angle AEC$ and $\angle CAE$ are complementary, as are $\angle CAD$ and $\angle CAE$. This equates as follows:

$$90^o = \angle AEC + \angle CAE = \angle CAD + \angle CAE$$
$$\angle AEC + \angle CAE = 3\theta + \angle CAE$$

Or simply,

$$\angle AEC = 3\theta$$

Furthermore where,

$$\angle BAC + \angle BAD = \angle CAD$$
$$\theta + \angle BAD = 3\theta$$
$$\angle BAD = 2\theta$$

From the resulting *Figure 10* geometry, the lengths indicated below are rather easily determined.

$$\overline{AC} = \overline{CD} = \overline{AD} = \cos\theta$$

$$\overline{BC} = \sin\theta$$

$$\overline{AF} = \cos(2\theta)$$

$$\overline{BF} = \sin(2\theta)$$

$$\overline{\mathrm{EC}} = \overline{\mathrm{AC}}/\tan(3\theta) = \overline{\mathrm{AC}}/\zeta = \overline{\mathrm{AC}}/\sqrt{3}$$

$$\overline{\mathrm{AE}} = \overline{\mathrm{AC}}/\sin(3\theta) = \overline{\mathrm{AC}}/\eta = \overline{\mathrm{AC}}/(\sqrt{3}/2) = 2\overline{\mathrm{AC}}/\sqrt{3}$$

8.2. Second and Fourth Order Complex Quadratic Identities.

The *Straight Line Equation* for line $\overline{AC}$ is:

$$y = mx$$
$$= \tan(3\theta)x$$
$$= \zeta x$$

With line $\overline{EB}$ being perpendicular to line $\overline{AC}$, its *Linear Equation* is as follows:

$$y = -(1/m)x + b$$
$$= -(1/\tan(3\theta))x + \overline{AE}$$
$$= -x/\zeta + 2\overline{AC}/\sqrt{3}$$

Since *Point B* lies on line $\overline{EB}$:

$$\mathrm{y_B} = -\mathrm{x_B}/\zeta + 2\overline{\mathrm{AC}}/\sqrt{3}$$
$$\mathrm{y_B}^2 = \mathrm{x_B}^2/\zeta^2 - 4\overline{\mathrm{AC}}\mathrm{x_B}/\zeta^2 + 4\overline{\mathrm{AC}}^2/\zeta^2$$

TRUE SCANS

Where, via Pythagorean Theorem:

$$\overline{AB} = 1 = x_B{}^2 + y_B{}^2$$
$$= x_B{}^2 + (1/\zeta^2)(x_B{}^2 - 4\overline{AC}x_B + 4\overline{AC}^2)$$

Or,

$$\zeta^2 = \zeta^2 x_B{}^2 + x_B{}^2 - 4\overline{AC}x_B + 4\overline{AC}^2$$

$$3 = (1+\zeta^2)x_B{}^2 - 4\overline{AC}x_B + 4\overline{AC}^2$$
$$= 4(x_B{}^2 - \overline{AC}x_B + \overline{AC}^2)$$

$$3/4 = x_B{}^2 - \overline{AC}x_B + \overline{AC}^2$$

Via substitution of one side of the equilateral triangle ACD for another, this results in the following primary relationship:

Equation 27. *Complex Quadratic Relationship.*

$$3/4 = x_B{}^2 - \overline{AD}x_B + \overline{AD}^2$$

Where, the above represents a *Complex Quadratic Equation* in the two unknowns, X_B and $\overline{AD}$.

An associated *transform* is derived easily by realizing that:

$$x_B = \cos(2\theta)$$
$$= 2\cos^2\theta - 1$$
$$= 2\overline{AD}^2 - 1$$

Whereby, *Equation 27* may be reformatted as follows:

$$3/4 = x_B{}^2 - \overline{AD}x_B + \overline{AD}^2$$
$$= (2\overline{AD}^2 - 1)^2 - \overline{AD}(2\overline{AD}^2 - 1) + \overline{AD}^2$$
$$= (4\overline{AD}^4 - 4\overline{AD}^2 + 1) - 2\overline{AD}^3 + \overline{AD} + \overline{AD}^2$$
$$= 4\overline{AD}^4 - 2\overline{AD}^3 - 3\overline{AD}^2 + \overline{AD} + 1$$

The resulting *Associated Quartic Equation* is as follows:

Equation 28. *Associated Quartic Relationship.*

$$4\overline{AD}^4 - 2\overline{AD}^3 - 3\overline{AD}^2 + \overline{AD} + 1/4 = 0$$

TRUE SCANS

8.3. Identities when *Each Equation Coefficient* Sums to Zero.

Invoking *Equation 1 [Ref. Table 12]*:

$$\cos^3\theta = 3/4\cos\theta + \tau/4$$

Hence, by substitution of $\overline{AD}$ for $\cos\theta$:

$$\overline{AD}^3 = (3/4)(\overline{AD}) + \tau/4$$

From *Equation 28*,

$$4\,\overline{AD}^4 - 2\,\overline{AD}^3 - 3\,\overline{AD}^2 + \overline{AD} + 1/4 = 0$$

$$(4\,\overline{AD} - 2\,)\overline{AD}^3 - 3\,\overline{AD}^2 + \overline{AD} + 1/4 = 0$$

Via further substitution, since $\tau = \cos(3\theta) = \cos 60^o = 1/2$:

$$(4\,\overline{AD} - 2\,)(\frac{3\overline{AD} + \tau}{4}) - 3\,\overline{AD}^2 + \overline{AD} + 1/4 = 0$$

$$[3\,\overline{AD}^2 + (1/2 - 3/2)\overline{AD} - 1/4] - 3\,\overline{AD}^2 + \overline{AD} + 1/4 = 0$$

$$(3 - 3)\,\overline{AD}^2 + (1-1)\overline{AD} + (1/4 - 1/4) = 0$$

This results in a <u>*numerical identity*</u> such that *each of the coefficients* of the equation sum to zero. As such, it provides ***no*** quantitative indication as to the actual length of side $\overline{AD}$.

8.4. Identities when the *Summation of Respective Equation Terms* Equate.

Lines $\overline{AC}$ and $\overline{EB}$ intersect at point C. Therefore, their respective equations, derived above, may be combined as follows:

$$y_C = \zeta x_C = -x_C/\zeta + 2\overline{AC}/\sqrt{3}$$

$$\zeta\frac{\overline{AD}}{2} = -\frac{\overline{AD}}{2\zeta} + \frac{2\overline{AD}}{\sqrt{3}}$$

$$\sqrt{3}\frac{\overline{AD}}{2} = -\frac{\overline{AD}}{2\sqrt{3}} + \frac{2\overline{AD}}{\sqrt{3}}$$

Which, by dividing thru by $\overline{AD}$, and rearranging gives:

$$\frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}} - \frac{1}{2\sqrt{3}}$$

Multiplying thru by $2\sqrt{3}$ finally yields

$$3 = 2(2) - 1$$
$$= 3$$

This results in a ***numerical identity*** without mentioning length $\overline{AD}$ which *drops out* (i.e.; has been *canceled out*)

TRUE SCANS

8.5. Mathematical Closure.

Equation 27 relates $\overline{AD}$ to a *second unknown quantity* x_B. As depicted, it qualifies as a *Complex Quadratic Equation*, thereby affording unique sets or *families of solutions*.

The format of this *Complex Quadratic Equation* is such that when standing alone (i.e.; without permitting it to become *associated* with any secondary, independent equation having the potential to mathematically support it), possible values can be ascribed to *either* of its *dual unknown quantities* in order to determine various sets of *solutions*.

The process of ***mathematical closure*** consists of ascribing all possible values to *either* of such *dual unknown quantities* in order to determine a complete set of respective solution pairs.
In other words, respective values for x_B can be readily determined, via *Quadratic Formula*, for various particular values of length $\overline{AD}$ that repetitively, independently are applied.

Such solutions do meet all conditions imposed upon them by *Equation 27*. However, a full-blown *mathematical closure* is necessary to guarantee that a particular value of length $\overline{AD} = \cos 20^o$ is applied, thereby enabling *Equation 27* to actually pertain to *Figure 10*.

This signifies that *Complex Quadratic Equations*, by their nature introduce an aspect of *mathematical uncertainty*.

Such *uncertainty* is elucidated upon below by means of an analysis which examines only two values for $\overline{AD}$, both of which invariably satisfy *Equation 27*. However only the later value meets the additional *Figure 10* imposed criterion that:

$$\overline{AD} = \cos 20^o$$

First, when $\overline{AD}$ is set equal to ½, *Equation 27* may be ***resolved*** as follows:

$$3/4 = x_B{}^2 - \overline{AD}x_B + \overline{AD}^2$$

$$3/4 = x_B{}^2 - (1/2)x_B + 1/4$$

Then,

$$x_B{}^2 - 1/2x_B - 1/2 = 0$$

Or by applying the *Quadratic Formula*,

$$x_B = [-b \pm \sqrt{b^2 - 4ac}]/2a$$
$$= [1/2 \pm \sqrt{1/4 - 4(1)(-1/2)}]/2(1)$$
$$= [1/2 \pm \sqrt{1/4 + 2)}]/2(1)$$
$$= [1/2 \pm \sqrt{9/4)}]/2$$
$$= [1/2 \pm 3/2]/2$$
$$= 1, - ½$$

For $\overline{AD}$ = ½, it is verified by substitution that these above roots do satisfy *Equation 27*, as follows:

$$3/4 = x_B{}^2 - \overline{AD}x_B + \overline{AD}^2$$
$$= x_B{}^2 - (1/2)x_B + 1/4$$
$$= 1^2 - (1/2)(1) + (1/2)^2$$
$$= 1 - 1/2 + 1/4$$
$$= 3/4$$

And,

$$3/4 = x_B{}^2 - (1/2)x_B + 1/4$$
$$= (-1/2)^2 - 1/2(-1/2) + (1/2)^2$$
$$= 1/4 + 1/4 + 1/4$$
$$= 3/4$$

Secondly, *Equation 27* also may be ***independently resolved*** for the exact condition when $\overline{AD} = \cos\theta = \cos 20^o$ as follows:

$$3/4 = x_B{}^2 - \overline{AD}x_B + \overline{AD}^2$$
$$3/4 = x_B{}^2 - (0.93969262)x_B + (0.93969262)^2$$

Then,

$$x_B{}^2 - (0.93969262)x_B + 0.133022221 = 0$$

Or via *Quadratic Formula*,

$$x_B = [-b \pm \sqrt{b^2 - 4ac}]/2a$$
$$= [0.93969262 \pm \sqrt{(-0.93969262)^2 - 4(1)(0.133022221)}]/2(1)$$
$$= [0.93969262 \pm \sqrt{0.883022221 - 0.532088884}]/2(1)$$
$$= [0.93969262 \pm \sqrt{0.350933337}]/2(1)$$
$$= [0.93969262 \pm 0.592396267]/2$$
$$= 0.766044443; 0.173648176$$
$$= \cos 40^o; \cos 80^o$$
$$= \cos 2\theta; \cos 4\theta$$

TRUE SCANS

For $\overline{AD} = \cos 20^o$, it is verified by substitution that these above roots do satisfy *Equation 27*, as follows:

$$3/4 = x_B{}^2 - \overline{AD}x_B + \overline{AD}^2$$
$$= x_B{}^2 - (0.93969262)x_B + (0.93969262)^2$$
$$= (0.766044443)^2 - (0.93969262)(0.766044443) + 0.883022221$$
$$= 0.586824088 - 0.719846309 + 0.883022221$$
$$= 3/4$$

And,

$$3/4 = x_B{}^2 - (0.93969262)x_B + (0.93969262)^2$$
$$= (0.17648176)^2 - (0.93969262)(0.17648176) + (0.93969262)^2$$
$$= 0.030153689 - 0.163175911 + 0.883022221$$
$$= 3/4$$

In conclusion, the two different values for $\overline{AD}$ applied above *both clearly satisfy Equation 27*. However, *Equation 27* ***doesn't know*** which of these should be applied in order to *satisfy* the additional constraints imposed by *Figure 10*.

Were this *two solution analysis* instead to become expanded to *address* **all values** that could possibly be assigned to length $\overline{AD}$, then a *mathematical closure*, or *full-blown* resolution of *Complex Quadratic Equation 27* naturally would result.

Moreover, the application of *mathematical closure* is not recommended in all cases. For example, since *Complex Cubic Equation 3* shown below exhibits *one root set* consisting of *three roots* for every distinct value of ζ that exists, such analysis would continue indefinitely:

$$\tan^3\theta = 3\tan - \zeta(1 - 3\tan^2\theta) \qquad \textit{[Ref. Equation 3]}$$

Although Equation 3 cannot determine which particular value of ζ its assessor is seeking - simply because *an inexhaustible number of root sets are afforded* which satisfy this equation, once a particular value becomes assigned to ζ, such as √3, only one *root set applies*, such that:

$$\tan^3\theta = 3\tan - \sqrt{3}(1 - 3\tan^2\theta)$$

[Ref. Equation 24] when $z_1 = \tan\theta$ *;*

[Ref. Equation 3] when $\zeta = \tan 60^o = \sqrt{3}$

Such that,

$$z_1; z_2; z_3 = \tan\theta; \tan(\theta + 120^o); \tan(\theta + 240^o) = \tan 20^o; \tan 140^o; \tan 260^o$$

SECTION 9. *NUMBER THEORY IMPLICATION.*

In order to *advance Number Theory state-of-the-art*, an attempt is made to explain the very existence for dissimilar *equation formats*, and the reason why *diversity* exists between them.

To this end, a comparison is conducted between *Quadratic and Cubic Equation formats* which reveals that:

a) Each exhibits a *mathematical structure* that actually is quite different in nature from the other;
b) Each functions in a *diverse* manner; and
c) Each exists for it own unique reason!

9.1. Rationally-based and Cubic Irrational Number Classifications.

<u>All</u> *real numbers* can be categorized either as *rationally-based* or *cubic irrational*, where:

Rationally-based numbers consist of:

a. *All rational numbers; and*
b. *Quadratic irrational numbers* such as $17\sqrt{35\sqrt{7/1025}}$ which are comprised of the magnitudes of all lengths which can be geometrically constructed from a given length of unity other than those which are of rational value. When algebraically expressed, they must exhibit at least one square root radical sign. However, *quadratic irrational numbers* cannot feature any radical sign which is a multiple of three, such as a cube root or even possibly an eighty-first root, because such values cannot be determined by means of applying successive Quadratic Formulas that are permitted to operate only upon either rational numbers and/or quadratic equation root values, as might become determined by such method; and

Cubic irrational numbers consist of <u>all</u> other real numbers that *cannot* be classified as *rationally-based*.

The *rationally-based number* classification should be viewed as a set of *real numbers* which includes all possible *Euclidean* determinations that can be ***geometrically constructed*** from a given, arbitrary length of unity.

It *collates* a disparate assortment of *lengths* together whose magnitudes all are of *rational* and *quadratic irrational values*, like $4+(32/62)\sqrt{5}+17\sqrt{35\sqrt{7/1025}}$, whose *individual terms* consist specifically of:

1) *Rational numbers* – represented as the *ratio* between two lengths whose magnitudes are integer values and portrayed as follows:

$$x_1 = \frac{\Delta}{2a} = \frac{x_1}{1}$$

The *mathematic division* represented above described by Δ/2a identifies a length, x_1, that is determined via ***geometric construction*** performed in accordance with the *Euclidean Mapping Process* specified in *Section 2.3*, where:

a) Lengths *Δ* and *2a*, each representing *integer values*, are ***geometrically constructed*** via sole *straightedge* and *compass* using an arbitrary, assigned length of *unity* as a basis; and

b) Rational *length* x_1 is identified as the horizontal offset measured from the right side of the rectangle to the point where the diagonal line intersects the horizontal line which exhibits a height of unity *(Ref. Figure 2)*.

Hence, all *rational numbers* are *Euclidean*!

In other words, each and every one can be ***geometrically constructed*** from an arbitrary length which is to be designated as one unit long via only a *straightedge* and *compass*; and

2) *Quadratic irrational numbers* – represented as magnitudes of lengths other than those which are of rational value which can be geometrically constructed from a given length of unity.

Although *rational values* can become *transformed* into quadratic *irrational values* via *Pythagorean Theorem*, it nevertheless remains possible to measure straight line lengths which exhibit such magnitudes, as well as to replicate them from a given, arbitrary length of unity.

All that needs to be known in order to geometrically construct a square root that is indicative of a *quadratic irrational number* is that upon drawing a right triangle whose sides become algebraically expressed as *a* and *b*, the altitude extending to its hypotenuse, *c*, will divide such base into two segments denoted respectively as *s* and (*c* – *s*). [For example, upon viewing *Figure 1*, *t*, amounting in length to straight line $\overline{AB}$ could be designated as side *a*, straight line $\overline{BC}$ could be designated as side *b*, straight line $\overline{AC}$ could be designated as hypotenuse, *c*, and straight line section $\overline{CE}$ could be designated as segment *s*.] Hence, due to three similar right triangles which thereby become described in such manner, two residing inside of such larger initially drawn right triangle, a trigonometric relationship of the form sin θ = b/c = s/b thereby could be established. In that the proportion b/c therein identifies sides belonging to such larger right triangle, the proportion s/b would apply to corresponding sides belonging to the smaller right triangle whose hypotenuse is of length b. By multiplying each side of such resulting equation by the factor bc, the equality $b^2 = cs$ becomes obtained. Then by taking the square root of each side, it becomes apparent that b = $\sqrt{cs}$. As various rational values become substituted for c and s therein, the length of side b of such larger right triangle thereby would assume different square root magnitudes. So, if it were intended to geometrically construct side b so that it amounts to $\sqrt{3}$ units in length, a right triangle could be drawn whose hypotenuse, c, amounts to 3 units in length such that the altitude which lies perpendicular to it would reside a distance away from either of its ends a total of one unit of measurement; thereby setting the value of s to be one unit long. Accordingly, the value of length *b* would amount to $\sqrt{cs} = \sqrt{3(1)} = \sqrt{3}$ units in overall length. In such very same manner, a fourth root of 3, as amounting to the square root of $\sqrt{3}$ and algebraically expressed as $3^{1/4} = (3^{1/2})^{1/2} = \sqrt{3^{1/2}} = \sqrt{\sqrt{3}}$, thereafter could be geometrically constructed, merely by means of drawing another right triangle which this time instead exhibits dimensions of c= $\sqrt{3}$ and s = 1, such that $b = \sqrt{cs} = \sqrt{\sqrt{3}(1)} = \sqrt{\sqrt{3}}.$

<u>In conclusion</u>:

- ***Rationally-based numbers*** comprise the magnitudes of *all* lengths which can be geometrically constructed from a given length of unity.
- ***Cubic irrational numbers*** comprise all other *real numbers*; specifically accounting for the magnitudes of *all* lengths which <u>cannot</u> be geometrically constructed from a given, arbitrary length of unity.

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9.2. The *Cubic Equation* Distinction.

Table 13, Table 14, and Table 15 present various equations which express *mathematical combinations* of *trigonometric, cubic irrational number roots* on their *right-hand sides* that actually can be collated into *rationally-based numerical results* which appear on their respective *left-hand sides*.

This is easily demonstrated for the specific case when 3θ assumes the value of 60°, whereby respective *left-hand terms* equate to *rationally-based* values of $-\tau/4 = -1/8$, $\eta/4 = \sqrt{3}/8$, $\zeta = \sqrt{3}$, zero, zero, $-3\zeta = -3\sqrt{3}$, -3/4, -3/4 and -3.

The tables depict ***product***, ***summation***, and ***summation of paired product*** breakdowns as follows:

- A *known, or given discrete rationally-based value* equals the ***product*** of three distinct, but linked, *trigonometric, cubic irrational number roots* as follows:

Table 13. The Product of Three Roots Equals a Known Value.

$$-\frac{\tau}{4} = -x_1 x_2 x_3$$ *[Ref. Equation 5]*

$$\frac{\eta}{4} = -y_1 y_2 y_3$$ *[Ref. Equation 8]*

$$\zeta = -z_1 z_2 z_3$$ *[Ref. Equation 11]*

- A *known, or given discrete rationally-based value* equals the ***summation*** of three distinct, but linked, *trigonometric, cubic irrational number roots* as follows:

Table 14. The Summation of Three Roots Equals a Known Value.

$$0 = x_1 + x_2 + x_3$$ *[Ref. Equation 6]*

$$0 = y_1 + y_2 + y_3$$ *[Ref. Equation 9]*

$$-3\zeta = -(z_1 + z_2 + z_3)$$ *[Ref. Equation 12]*

- A *known, or given discrete rationally-based value* equals the ***summation of paired products*** of three distinct, but linked, *trigonometric, cubic irrational number roots* as follows:

Table 15. The Sum of Paired Products of Three Roots Equals a Known Value.

$$-\frac{3}{4} = x_1 x_2 + x_1 x_3 + x_2 x_3$$ *[Ref. Equation 7]*

$$-3/4 = y_1 y_2 + y_1 y_3 + y_2 y_3$$ *[Ref. Equation 10]*

$$-3 = z_1 z_2 + z_1 z_3 + z_2 z_3$$ *[Ref. Equation 13]*

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Rationally-based quantities listed on the *left-hand sides* of the above equations represent respective ***coefficients*** for the following three *Cubic Equations*.

$$\cos^3\theta - (3/4)\cos\theta - \tau/4 = 0 \quad \textit{[Ref. Equation 1]}$$

$$\sin^3\theta - (3/4)\sin\theta + \eta/4 = 0 \quad \textit{[Ref. Equation 2]}$$

$$\tan^3\theta - 3\zeta\tan^2\theta - 3\tan\theta + \zeta = 0 \quad \textit{[Ref. Equation 3]}$$

Moreover, *trigonometric, cubic irrational quantities* afforded on the *left-hand sides* of these three equations constitute their respective *root structures*.

Lastly, notice that these three *Cubic Equations* are comprised entirely, or solely of *rationally-based* coefficients.

Equation 3 is validated below for the specific case when 3θ again assumes the value of 60° such that:

$\zeta = \sqrt{3}$

$$\tan^3\theta - 3\zeta\tan^2\theta - 3\tan\theta + \zeta = 0 \quad \textit{[Ref. Equation 3]}$$

$$\tan^3\theta - 3\sqrt{3}\tan^2\theta - 3\tan\theta + \sqrt{3} = 0$$

From *Section 2.3*,

- z_1 = tan θ = tan 20° = 0.363970234
- z_2 = tan (θ + 120°) = tan 140° = -0.839099631
- z_3 = tan (θ + 240°) = tan 260° = 5.67128812

Where,

$$(\tan\theta - z_1)(\tan\theta - z_2)(\tan\theta - z_3) = 0$$

$$[\tan^2\theta - (z_1 + z_2)\tan\theta + z_1 z_2] - (\tan\theta - z_3) = 0$$

$$\tan^3\theta - (z_1 + z_2 + z_3)\tan^2\theta + (z_1 z_2 + z_1 z_3 + z_2 z_3)\tan\theta - z_1 z_2 z_3 = 0$$

Equating respective *coefficient terms* gives:

$$-3\sqrt{3} = -(z_1 + z_2 + z_3)$$

$$-3 = z_1 z_2 + z_1 z_3 + z_2 z_3$$

$$\sqrt{3} = -z_1 z_2 z_3$$

Trigonometric, cubic irrational root set values then are substituted to *confirm* that, despite their various *arrangements*, they nevertheless *mathematically comprise* each of the *given rational coefficients* as follows:

Where,

$$3\zeta = z_1 + z_2 + z_3 \quad \textit{[Ref. Equation 12]}$$

$$= 0.363970234 - 0.839099631 + 5.67128812$$

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$$= 3\sqrt{3}$$

$$-3 = z_1z_2 + z_1z_3 + z_2z_3 \quad \textit{[Ref. Equation 13]}$$

$$\begin{aligned}
&= z_1(z_2 + z_3) + z_2z_3 \\
&= (0.363970234)(-0.839099631 + 5.67128812) + (-0.839099631)(5.67128812) \\
&= (0.363970234)(4.832182188) + (-0.839099631)(5.67128812) \\
&= 1.758770483 - 4.758770483 \\
&= -3
\end{aligned}$$

$$-\zeta = z_1z_2z_3 \quad \textit{[Ref. Equation 11]}$$

$$\begin{aligned}
&= (0.363970234)(-0.839099631)(5.67128812) \\
&= -\sqrt{3}
\end{aligned}$$

Another example is afforded below whereby *coefficients* of the following given *Cubic Equation* consist solely of *rational numbers*:

$$u^3 - 4u^2 + 3u - \frac{1}{4} = 0$$

Such that

$$(u - u_R)(u - u_S)(u - u_T) = 0$$

$$u^3 - (u_R + u_S + u_T)u^2 + (u_Ru_S + u_Ru_T + u_Su_T)u - u_Ru_Su_T = 0$$

Equating respective *coefficient terms* gives:

$$-4 = -(u_R + u_S + u_T)$$

$$3 = u_Ru_S + u_Ru_T + u_Su_T$$

$$-\frac{1}{4} = -u_Ru_Su_T$$

The respective *trigonometric, cubic irrational root set values* for this given *Cubic Equation*, as determined via later analysis *(Ref. Section 13.3.4)*, are specified as follows:

$$u_R; u_S; u_T = +3.040302198, +0.864590763 + 0.09510704$$

Then, by substitution:

$$\begin{aligned}
-4 &= -(u_R + u_S + u_T) \\
&= -(3.040302198 + 0.864590763 + 0.09510704) \\
&= -4
\end{aligned}$$

$$\begin{aligned}
3 &= u_Ru_S + u_Ru_T + u_Su_T \\
&= u_R(u_S + u_T) + u_Su_T \\
&= 3.040302198(0.864590763 + 0.09510704) + (0.864590763)(0.09510704) \\
&= 3.040302198(0.959697803) + 0.082228668 \\
&= 3
\end{aligned}$$

$$-\frac{1}{4} = -u_R u_S u_T$$
$$= -(3.040302198)(0.864590763)(0.09510704)$$
$$= -(3.040302198)(0.082228668)$$
$$= -0.25$$

This unique capability to characterize *trigonometric, cubic irrational roots* in terms of sole *rationally-based coefficients* is reserved only for *Cubic Equation formats.*

Furthermore, *Quadratic Equation formats* do not possess this ability, simply because they require at least one *cubic irrational coefficient* to be present in order to produce a *trigonometric, cubic irrational root pair*.

This above assertion is validated by considering either the following *algebraic Quadratic Equation* or its *associated function*:

$$ax^2 + bx + c = 0$$

Or,

$$ax^2 + bx + c = y$$
$$ax^2 + bx + (c - y) = 0$$
$$ax^2 + bx + c' = 0$$

Dividing thru by the *first term coefficient* produces:

$$x^2 + \frac{b}{a}x + \frac{c'}{a} = 0$$

Or,

$$x^2 + b'x + c'' = 0$$

Now, the *root pair* $x_1;x_2$ belonging to the derived equation above can be determined, or viewed in terms of the famous *Quadratic Formula* as follows:

$$x_1;x_2 = \frac{1}{2}(-b' \pm \sqrt{b'^2 - 4c''})$$

For particular circumstances when both *coefficients b'* and c'' are *rationally-based*, it is obvious that such $x_1;x_2$ *root pair* also must remain *rationally-based*.

Conversely, for the $x_1;x_2$ *root pair* to be *cubic irrational*, it then would have to violate the *redefined definition* of *cubic irrational numbers*, as postulated in *Section 9.1*.

9.3. *Cubic Equation Uniqueness Theorem*.

Based upon the above premise that:

- *Cubic Equations* possess an ***innate ability*** to characterize *trigonometric, cubic irrational roots* in terms of sole *rationally-based coefficients*
- *Linear* or *Quadratic Equation formats* cannot in any manner duplicate this capability

It could be argued that *Cubic Equation formats* pose a complete *demarcation* from their *Linear* and *Quadratic Equation counterparts*. This is because they *must exist* as separate *mathematical entities, independent or completely apart* from *Quadratic Equation formats*, in order *allow* for a unique correlation between *rationally-based coefficients* and their associated *cubic irrational root sets*.

Such contention prefers an ***extraordinary implication*** upon *Number Theory* by suggesting that equations well might assume unque formats in order *to account for the very numerical representations included therein*.

This gives rise to a new ***Cubic Equation Uniqueness Theorem*** as described below. It applies exclusively to *equation formats* of ***singular unknown*** quantity *(Ref. Section 2.2)*:

> ***Only Cubic Equations allow solely rationally-based numerical coefficients to co-exist with root sets comprised of cubic irrational numbers.***

This theorem in no way disputes, or contradicts the fact that *cubic irrational root pairs* can, and do exist within *Quadratic Equation formats*.

What is very interesting, predicated upon what was deduced above, is that the only way this can occur is when *coefficients b′* and/or *c″* also are *cubic irrational*.

As such, a ***corollary*** to the *Cubic Equation Uniqueness Theorem* appears below:

> ***Cubic irrational root pairs* which appear in *Parabolic Equations* or their *associated functions* require *supporting cubic irrational coefficients*.**

The above determinations do <u>not</u> address *Complex Quadratic* and *Complex Cubic Equations/Functions*. Hence, all of the pitfalls normally experienced when dealing with *identities* resulting from associated *reductions* of such *equations* are thereby avoided!

The third, fourth, and fifth rows of the *logic diagram* specified below apply to the above *corollary*:

CASE	x	b'	x^2	b'x	c''	CUBIC IRRATIONAL PAIRING
I	Rationally-based	Rationally-based	Rationally-based	Rationally-based	Rationally-based	*N/A**
II	Rationally-based	*Cubic irrational*	Rationally-based	*Cubic irrational*	*Cubic irrational*	*b' and c''*
III	*Cubic irrational*	Rationally-based	*Cubic irrational*	*Cubic irrational*	*Cubic irrational*	*x and c''*
IV	*Cubic irrational*	*Cubic irrational*	*Cubic irrational*	*Cubic irrational*	*Cubic irrational*	*x, b', and c''*
V	*Cubic irrational*	*Cubic irrational*	*Cubic irrational*	*Cubic irrational*	Rationally-based	*x, and b'*

*Equation contains no *cubic irrational numbers*

An example for each case is presented below:

CASE	x	b'	x^2	b'x	c''	$x^2 + b'x + c'' = 0$ SUM
I	$4+\sqrt{5}$	$4-\sqrt{5}$	$21+8\sqrt{5}$	11	$-(32+8\sqrt{5})$	0
II	2	0.939692621	4	1.879385242	-5.87938524	0
III	0.939692621	$4+\sqrt{5}$	0.883022222	5.859987061	-6.743009283	0
IV	0.939692621	0.766044443	0.883022222	0.71984631	-1.60286853	0
V	5.67128182	-5.142300877	32.16343748	-29.16343748	-3	0

SECTION 10. *THE UCTRE; AND INTRODUCTION OF EQUATION SUB-ELEMENTS.*

This section presents a novel *missing link transform*, hereinafter referred to as the ***Unified Cubic Trigonometric Reduction Equation (UCTRE)***.

Being of the overall form $ax^2 + bx + c = 0$, *root pairs* easily can be determined by *mathematical assessment* conducted solely upon *UCTRE* inherent *coefficients* via the *Quadratic Formula* as follows:

$$x = [-b \pm \sqrt{b^2 - 4ac}]/2a$$

Of extreme practical importance, *UCTRE coefficients* also can be shown to house ***vital information*** which actually characterizes *root set structures* evident within higher order *Cubic Equations* which they are associated with.

Such *vital information* manifests itself in the form of ***RST Terminology***, otherwise hereinafter deemed ***Equation Sub-elements***.

RST Terminology consists of a set of ***factors*** which relate $\tan\theta$ to respective *Cubic Equations roots* as follows:

$$z_R = R\tan\theta$$

$$z_S = S\tan\theta$$

$$z_T = T\tan\theta$$

Accordingly, the *UCTRE* functions as a *conduit* which thereby enables *RST Terminology* embedded within higher order *Cubic Equations* to be *dispensed* directly into it. This occurs because the *UCTRE*, in itself, actually performs as a *reduced Quadratic Equation*.

During such *reduction*, an *internal linkage capability* apparently is at work which relates *Cubic Equation root set structures* to information that is contained in resulting *lower order UCTRE coefficients.*

Derivation:

Since the roots for *Equation 3* are *three trigonometric values* which characterize angles spaced out 120° apart *(Ref. Section 2.4.3)*:

- $z_1 = \tan\theta = \tan\theta_1$
- $z_2 = \tan(\theta + 120°) = \tan\theta_2 = (\tan\theta - \sqrt{3})/(1+\sqrt{3}\tan\theta)$
- $z_3 = \tan(\theta + 240°) = \tan\theta_3 = (\tan\theta + \sqrt{3})/(1-\sqrt{3}\tan\theta)$

$\Sigma = 3\theta + 360°$
$= 3\theta$

It's plausible to let:

$$z_R = R\tan\theta = \tan\theta_R = \tan(\theta+\alpha)$$
$$z_S = S\tan\theta = \tan\theta_S = \tan(\theta+\beta)$$
$$z_T = T\tan\theta = \tan\theta_T = \tan(\theta+\gamma)$$

Where,

$$\theta_R + \theta_S + \theta_T = 3\theta$$
$$(\theta+\alpha)+(\theta+\beta)+(\theta+\gamma) = 3\theta$$
$$\theta+\gamma = 3\theta - [(\theta+\alpha)+(\theta+\beta)]$$

Then,

$$z_T = T\tan\theta = \tan(\theta+\gamma) = \tan\{3\theta - [(\theta+\alpha)+(\theta+\beta)]\}$$
$$= \frac{\zeta - \tan[(\theta+\alpha)+(\theta+\beta)]}{1+\zeta\tan[(\theta+\alpha)+(\theta+\beta)]}$$
$$= \frac{\zeta - \dfrac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{1-\tan(\theta+\alpha)\tan(\theta+\beta)}}{1+\zeta\dfrac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{1-\tan(\theta+\alpha)\tan(\theta+\beta)}}$$
$$= \frac{\zeta - \dfrac{R\tan\theta + S\tan\theta}{1-RS\tan^2\theta}}{1+\zeta\dfrac{R\tan\theta + S\tan\theta}{1-RS\tan^2\theta}}$$
$$T\tan\theta = \frac{\zeta - \zeta RS\tan^2\theta - (R+S)\tan\theta}{1-RS\tan^2\theta + \zeta(R+S)\tan\theta}$$
$$T\tan\theta - RST\tan^3\theta + \zeta T(R+S)\tan^2\theta = \zeta - \zeta RS\tan^2\theta - (R+S)\tan\theta$$

Substituting $3\tan\theta - \zeta(1-3\tan^2\theta)$ for $\tan^3\theta$ *(Ref. Equation 3)* gives:

$$T\tan\theta - RST[3\tan\theta - \zeta(1-3\tan^2\theta)] + \zeta T(R+S)\tan^2\theta = \zeta - \zeta RS\tan^2\theta - (R+S)\tan\theta$$
$$(R+S+T)\tan\theta - 3RST\tan\theta + \zeta RST - 3\zeta RST\tan^2\theta + \zeta(RS+RT+ST)\tan^2\theta - \zeta = 0$$

Which simplifies to:

Equation 29. ***The Unified Cubic Trigonometric Reduction Equation.***

$$\zeta(RST-1) + [(R+S+T)-3RST]\tan\theta + \zeta[(RS+RT+ST)-3RST]\tan^2\theta = 0$$

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Equation 29, re-written as follows, is of the following *Quadratic* form:

$$ax^2 \quad + \quad bx \quad + \quad c \quad = 0$$

$$\zeta[(RS+RT+ST)-3RST]\tan^2\theta+[(R+S+T)-3RST]\tan\theta+\zeta(RST-1)=0$$

Where,

$$a=\zeta(RS+RT+ST-3RST)$$
$$b=R+S+T-3RST$$
$$c=\zeta(RST-1)$$
$$x=\tan\theta$$

As such, *Equation 29*, exemplifies a *Quadratic Equation* whose coefficients are represented in terms of combinations of R, S, T, and ζ.

Hence, the *Quadratic Formula* applies as follows:

$$\mathrm{x}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\tan\theta=\frac{[3RST-(R+S+T)]\pm\sqrt{[(R+S+T)-3RST)]^2+4\zeta^2[3RST-(RS+RT+ST)](RST-1)}}{2\zeta(RS+RT+ST-3RST)}$$

Therefore, the *dual roots* for *Equation 29*, namely x_1 and x_2, may be represented in terms of combinations of R, S, T, and ζ. For any particular value of tan θ, many variations of *Equation 29* exist depending on the assignments of such coefficients.

As indicated below, R, S, and T may be multiplied respectively by tan θ in order to determine three roots z_R, z_S, and z_T for a *Cubic Equation* as follows.

- $z_R = R\tan\theta$
- $z_S = S\tan\theta$
- $z_T = T\tan\theta$

Moreover, *Equation 29 coefficients* express combinations of the respective *products*, *summations*, and *summations of the paired products* of ***Equation Sub-elements R, S, and T.***

When substituting the roots for *Equation 3* back into *Equation 29, it simply reverts back into Equation 3*, thereby verifying the math used during the *Equation 29* derivation, as follows:

__Verification__ (Reverse derivation)__:__

- $z_R = R \tan\theta$
- $z_S = S \tan\theta$
- $z_T = T \tan\theta$

  - $R = z_1 / \tan\theta$
  - $S = z_2 / \tan\theta$
  - $T = z_3 / \tan\theta$

But,

$$RST = \frac{z_1 z_2 z_3}{\tan^3\theta} = -\frac{\zeta}{\tan^3\theta}$$ *[Ref. Equation 11]*

$$3RST = \frac{3z_1 z_2 z_3}{\tan^3\theta} = -\frac{3\zeta}{\tan^3\theta}$$

$$R+S+T = \frac{z_1+z_2+z_3}{\tan\theta} = \frac{3\zeta}{\tan\theta}$$ *[Ref. Equation 12]*

$$RS+RT+ST = \frac{z_1 z_2 + z_1 z_3 + z_2 z_3}{\tan^2\theta} = -\frac{3}{\tan^2\theta}$$ *[Ref. Equation 13]*

$$\zeta(RST-1)+[(R+S+T)-3RST]\tan\theta+\zeta[(RS+RT+ST)-3RST]\tan^2\theta=0$$ *[Ref. Equation 29]*

$$\zeta(-\frac{\zeta}{\tan^3\theta}-1)+[(\frac{3\zeta}{\tan\theta})-\frac{-3\zeta}{\tan^3\theta}]\tan\theta+\zeta[(-\frac{3}{\tan^2\theta})-\frac{-3\zeta}{\tan^3\theta}]\tan^2\theta=0$$

$$\zeta(-\frac{\zeta}{\tan^3\theta}-\frac{\tan^3\theta}{\tan^3\theta})+[(\frac{3\zeta}{\tan\theta})\frac{\tan^2\theta}{\tan^2\theta}+\frac{3\zeta}{\tan^3\theta}]\tan\theta+\zeta[(-\frac{3}{\tan^2\theta})\frac{\tan\theta}{\tan\theta}+\frac{3\zeta}{\tan^3\theta}]\tan^2\theta=0$$

$$-\zeta(\zeta+\tan^3\theta)+3\zeta[\tan^2\theta+1]\tan\theta-3\zeta[\tan\theta-\zeta]\tan^2\theta=0$$

$$-\zeta^2-\zeta\tan^3\theta+3\zeta\tan^3\theta+3\zeta\tan\theta-3\zeta\tan^3\theta+3\zeta^2\tan^2\theta=0$$

$$-\zeta^2+3\zeta\tan\theta+3\zeta^2\tan^2\theta-\zeta\tan^3\theta+3\zeta\tan^3\theta-3\zeta\tan^3\theta=0$$

$$-\zeta^2+3\zeta\tan\theta+3\zeta^2\tan^2\theta-\zeta\tan^3\theta=0$$

$$-\zeta+3\tan\theta+3\zeta\tan^2\theta-\tan^3\theta=0$$

$$3\tan\theta-\zeta[1-3\tan^2\theta]=\tan^3\theta$$ *[Ref. Equation 3]*

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SECTION 11. *ASSOCIATED TRANSFORMS.*

This section develops transforms which are considered to be associated with the conduct of the *Unified Cubic Trigonometric Reduction Equation*.

11.1. The Simplified Unified Cubic Trigonometric Reduction Equation.

The Unified Cubic *Trigonometric Reduction Equation* may be simplified further by applying the following identities:

$$B = -(R + S + T)$$

$$C = RS + RT + ST$$

$$D = -RST$$

Then *Equation 29* further reduces as follows,

$$\zeta(RST - 1) + [(R + S + T) - 3RST]\tan\theta + \zeta[(RS + RT + ST) - 3RST]\tan^2\theta = 0$$ *[Ref. Equation 29]*

$$-\zeta(D + 1) - (B - 3D)\tan\theta + \zeta(C + 3D)\tan^2\theta = 0$$

Or,

Equation 30. ***The Simplified Unified Cubic Trigonometric Reduction Equation.***

$$\zeta(C + 3D)\tan^2\theta - (B - 3D)\tan\theta - \zeta(D + 1) = 0$$

Now, by applying the *Quadratic Formula* to *Equation 30*, a another simplified relationship for tan θ in terms of ζ , R, S, and T is realized as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\tan\theta = \frac{(B - 3D) \pm \sqrt{[-(B - 3D)]^2 + 4\zeta^2(C + 3D)(D + 1)}}{2\zeta(C + 3D)}$$

11.2. The Characteristic Cubic Equation.

Whereas R, S and T represent three *coefficients* from the *Unified Cubic Trigonometric Reduction Equation 29*, they also may be permitted to depict *roots* to a corresponding *Characteristic Cubic Equation*, derived as follows:

Where,

$$(q - R)(q - S)(q - T) = 0$$

$$[q^2 - (R + S)q + RS](q - T) = 0$$

$$q^3 - (R + S + T)q^2 + (RS + RT + ST)q - RST = 0$$

Now, because q can assume a value of *any* of the three roots R, S, or T above, the following equation results:

Equation 31. ***The Characteristic Cubic Equation.***

$$AR^3 + BR^2 + CR + D = 0$$

Furthermore, it may be easily noticed that as specific values for R, S, and T are interchanged with one another, the resulting values of B, C, and D, again notated below, are not impacted; i.e., they remain unaltered in value:

Six interchangeable combinations exist as follows, or three factorial (3!):

SPECIFIC "R" VALUE	**SPECIFIC "S" VALUE**	**SPECIFIC "T" VALUE**
R_1	S_1	T_1
R_1	T_1	S_1
S_1	R_1	T_1
S_1	T_1	R_1
T_1	R_1	S_1
T_1	S_1	R_1

As R assumes any of the values R_1, S_1, or T_1, then each of its respective S and T values can assume either of the combinations of the remaining values. That is to say, as R assumes a value of T_1 and S assumes a value of R_1, then T assumes a value of S_1; or as R assumes a value of T_1 and S assumes a value of S_1, T then assumes a value of R_1.

Accordingly,

$$
\begin{aligned}
B &= -(R+S+T) \\
&= -(R_1+S_1+T_1) \\
&= -(R_1+T_1+S_1) = -(R_1+S_1+T_1) \\
&= -(S_1+R_1+T_1) = -(R_1+S_1+T_1) \\
&= -(S_1+T_1+R_1) = -(R_1+S_1+T_1) \\
&= -(T_1+R_1+S_1) = -(R_1+S_1+T_1) \\
&= -(T_1+S_1+R_1) = -(R_1+S_1+T_1)
\end{aligned}
$$

Notice that the value for B above always remains exactly the same; i.e.; it is equal to the *summation* of R_1 plus S_1 plus T_1, regardless of what order is applied.

Likewise, the *sum of the paired products* is:

$$
\begin{aligned}
C &= RS+RT+ST \\
&= R_1S_1+R_1T_1+S_1T_1 \\
&= R_1T_1+R_1S_1+T_1S_1 = R_1T_1+R_1S_1+S_1T_1 = R_1S_1+R_1T_1+S_1T_1 \\
&= S_1R_1+S_1T_1+R_1T_1 = R_1S_1+S_1T_1+R_1T_1 = R_1S_1+R_1T_1+S_1T_1 \\
&= S_1T_1+S_1R_1+T_1R_1 = S_1T_1+R_1S_1+R_1T_1 = R_1S_1+R_1T_1+S_1T_1 \\
&= T_1R_1+T_1S_1+R_1S_1 = R_1T_1+S_1T_1+R_1S_1 = R_1S_1+R_1T_1+S_1T_1 \\
&= T_1S_1+T_1R_1+S_1R_1 = S_1T_1+R_1T_1+R_1S_1 = R_1S_1+R_1T_1+S_1T_1
\end{aligned}
$$

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Again, the value of C remains unaltered regardless of which combination is applied.

Lastly,

$$D = -RST$$
$$= -R_1S_1T_1$$
$$= -R_1T_1S_1 = -R_1S_1T_1$$
$$= -S_1R_1T_1 = -R_1S_1T_1$$
$$= -S_1T_1R_1 = -R_1S_1T_1$$
$$= -T_1R_1S_1 = -R_1S_1T_1$$
$$= -T_1S_1R_1 = -R_1S_1T_1$$

As indicated, the *product* of all *possible combinations* calculate to the same exact value of D, which is equal to $R_1S_1T_1$.

The above analysis can be interpreted to mean that for any given *Characteristic Cubic Equation (Ref. Equation 31)*, only one *Quadratic Equation* can result whose coefficients a, b, and c are as follows (Ref. *Equation 30*):

$$a = \zeta(C+3D)$$
$$b = -(B-3D)$$
$$c = -\zeta(D+1)$$

11.3. The Generalized Cubic Equation.

This section emphasizes that a unique *Generalized Cubic Equation* exists for each and every *Characteristic Cubic Equation*, derived as follows:

[Ref. Equation 31] $$AR^3 + BR^2 + CR + D = 0$$

For A = 1, multiplying thru by $\tan^3\theta$ yields,

$$R^3\tan^3\theta + BR^2\tan^3\theta + CR\tan^3\theta + D\tan^3\theta = 0$$
$$(R\tan\theta)^3 + (B\tan\theta)(R\tan\theta)^2 + (C\tan^2\theta)(R\tan\theta) + D\tan^3\theta = 0$$

Designating $R\tan\theta$ as z gives:

$$z^3 + (B\tan\theta)z^2 + (C\tan^2\theta)z + D\tan^3\theta = 0$$

Or,

Equation 32. ***The Generalized Cubic Equation.***

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$

Where,

$$\alpha = 1$$
$$\beta = B \tan \theta$$
$$\gamma = C \tan^2 \theta$$
$$\delta = D \tan^3 \theta$$

Moreover looking at the roots for *Equation 32*,

$$(z - z_R)(z - z_S)(z - z_T) = 0$$
$$[z^2 - (z_R + z_S)z + z_R z_S](z - z_T) = 0$$
$$z^3 - (z_R + z_S + z_T)z^2 + (z_R z_S + z_R z_T + z_S z_T)z - z_R z_S z_T = 0$$

But,

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

So, after comparing like coefficients:

$$\beta = -(z_R + z_S + z_T) = B \tan\theta$$
$$= -(R + S + T) \tan \theta$$
$$= -(R \tan \theta + S \tan \theta + T \tan \theta)$$

$$\gamma = z_R z_S + z_R z_T + z_S z_T = C \tan^2 \theta$$
$$= (RS + RT + ST) \tan^2 \theta$$
$$= (R \tan \theta) S \tan \theta + (R \tan \theta) T \tan \theta + (S \tan \theta) T \tan \theta$$

$$\delta = -z_R z_S z_T = D \tan^3 \theta$$
$$= -(RST) \tan^3 \theta$$
$$= -(R \tan \theta)(S \tan \theta)(T \tan \theta)$$

Therefore,

$$z_R = R \tan \theta$$
$$z_S = S \tan \theta$$
$$z_T = T \tan \theta$$

11.4.. Expression for S and T.

An expression for S and T is derived as follows:

Where,

$$B = -(R + S + T)$$
$$D = -RST$$

$$S + T = -(B + R)$$

Also,

$$S - T = (S + T) - 2T$$

TRUE SCANS

Such that,

$$(S-T)^2 = (S+T)^2 - 4T(S+T) + 4T^2$$
$$= [-(B+R)]^2 - 4ST - 4T^2 + 4T^2$$
$$= [-(B+R)]^2 - 4ST$$
$$= [-(B+R)]^2 - 4ST(\frac{-D}{RST})$$
$$= [-(B+R)]^2 + \frac{4D}{R}$$

Secondly,

$$\mathrm{S,T} = ½[(\mathrm{S+T}) \pm (\mathrm{S-T})]$$
$$= ½[(\mathrm{S+T}) \pm \sqrt{(S-T)^2}\,]$$

Equation 33. ***Expression for S and T.***

$$\mathrm{S,T} = ½[-(B+R) \pm \sqrt{(B+R)^2 + \frac{4D}{R}}\,]$$

Or,

$$\mathrm{S} = ½[-(B+R) + \sqrt{(B+R)^2 + \frac{4D}{R}}\,]$$
$$\mathrm{T} = ½[-(B+R) - \sqrt{(B+R)^2 + \frac{4D}{R}}\,]$$

11.5. Expression for R and (S + T).

An expression for R and (S + T) is derived as follows:

Where,

$$B = -(R+S+T)$$
$$C = RS + RT + ST$$
$$D = -RST$$

$$S + T = -(B+R)$$
$$RS + RT = -R(B+R)$$

$$RB = -R(R+S+T)$$
$$4RB = -4(R^2 + RS + RT)$$
$$= -4(R^2 + C - ST)$$
$$= -4(R^2 + C + \frac{D}{R})$$

$$\mathrm{R,(S+T)=R,\text{-}(B+R)}$$
$$=\tfrac{1}{2}(2\mathrm{R}),\tfrac{1}{2}(\text{-}2)(\mathrm{B+R})$$
$$=\tfrac{1}{2}[2\mathrm{R},\text{-}\,2(\mathrm{B+R})]$$
$$=\tfrac{1}{2}[\text{-}\mathrm{B}\pm(\mathrm{B+2R})]$$
$$=\tfrac{1}{2}[-B\pm\sqrt{(B+2R)^2}\,]$$
$$=\tfrac{1}{2}[-B\pm\sqrt{B^2+4RB+4R^2}\,]$$
$$=\tfrac{1}{2}[-B\pm\sqrt{B^2-4(R^2+C+\frac{D}{R})+4R^2}\,]$$

Or,

Equation 34. ***Expression for R and S + T.***

$$\mathrm{R,(S+T)}=\tfrac{1}{2}[-B\pm\sqrt{B^2-4(C+\frac{D}{R})}\,]$$

Or,

$$\mathrm{R,}=\tfrac{1}{2}[-B+\sqrt{B^2-4(C+\frac{D}{R})}\,]$$
$$\mathrm{S+T}=\tfrac{1}{2}[-B-\sqrt{B^2-4(C+\frac{D}{R})}\,]$$

11.6. Cubic Restitution Equation.

Now, it becomes possible to establish a relationship between tan θ in terms of ζ and the coefficients from the *Characteristic Cubic Equation 31* as follows:

Where $3\theta=\theta_R+\theta_S+\theta_T$,

$$z_R=\mathrm{R}\tan\theta=\tan\theta_R=\tan[3\theta-(\theta_S+\theta_T)]$$
$$z_S=\mathrm{S}\tan\theta=\tan\theta_S=\frac{\tan\theta}{2}[-(B+R)+\sqrt{(B+R)^2+\frac{4D}{R}}\,]$$
$$z_T=\mathrm{T}\tan\theta=\tan\theta_T=\frac{\tan\theta}{2}[-(B+R)-\sqrt{(B+R)^2+\frac{4D}{R}}\,]$$

Such that,

$$\tan\theta_S\tan\theta_T=\frac{\tan^2\theta}{4}\{[-(B+R)]^2-(B+R)^2-\frac{4D}{R}\}$$
$$=\frac{\tan^2\theta}{4}[(B+R)]^2-(B+R)^2-\frac{4D}{R}]$$
$$=-\frac{D\tan^2\theta}{R}$$

$$\tan\theta_S + \tan\theta_T = -(B+R)\tan\theta$$

$$\begin{aligned}
\tan\theta_R = R\tan\theta &= \tan[3\theta - (\theta_S + \theta_T)] \\
&= \frac{\tan(3\theta) - \tan(\theta_S + \theta_T)}{1 + \tan(3\theta)\tan(\theta_S + \theta_T)} \\
&= \frac{\zeta - (\dfrac{\tan\theta_S + \tan\theta_T}{1 - \tan\theta_S \tan\theta_T})}{1 + \zeta(\dfrac{\tan\theta_S + \tan\theta_T}{1 - \tan\theta_S \tan\theta_T})} \\
&= \frac{\zeta - \zeta\tan\theta_S \tan\theta_T - (\tan\theta_S + \tan\theta_T)}{1 - \tan\theta_S \tan\theta_T + \zeta(\tan\theta_S + \tan\theta_T)} \\
&= \frac{\zeta + \zeta(\dfrac{D\tan^2\theta}{R}) + (B+R)\tan\theta}{1 + (\dfrac{D\tan^2\theta}{R}) - \zeta(B+R)\tan\theta} \\
&= \frac{\zeta R + \zeta D\tan^2\theta + R(B+R)\tan\theta}{R + D\tan^2\theta - \zeta R(B+R)\tan\theta}
\end{aligned}$$

Cross multiplying yields,

$$R^2\tan\theta + DR\tan^3\theta - \zeta R^2(B+R)\tan^2\theta = \zeta R + \zeta D\tan^2\theta + R(B+R)\tan\theta$$

$$R^2\tan\theta + DR\tan^3\theta - \zeta R^2 B\tan^2\theta - \zeta R^3\tan^2\theta - \zeta R - \zeta D\tan^2\theta - R(B+R)\tan\theta = 0$$

Now, from the *Characteristic Cubic Equation* a relationship for AR^3 is determined as follows:

$$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$

$$AR^3 = -(BR^2 + CR + D)$$

Placing this result back into the expression notated above for A = 1 establishes the *Cubic Restitution Equation* as follows:

$$R^2\tan\theta + DR\tan^3\theta - \zeta R^2 B\tan^2\theta + \zeta[BR^2 + CR + D]\tan^2\theta - \zeta R - \zeta D\tan^2\theta - R(B+R)\tan\theta = 0$$

$$R^2\tan\theta + DR\tan^3\theta + \zeta(-BR^2 + BR^2 + CR + D)\tan^2\theta - \zeta R - \zeta D\tan^2\theta - R(B+R)\tan\theta = 0$$

$$R^2\tan\theta + DR\tan^3\theta + \zeta(CR + D)\tan^2\theta - \zeta R - \zeta D\tan^2\theta - R(B+R)\tan\theta = 0$$

$$DR\tan^3\theta + \zeta CR\tan^2\theta - \zeta R - RB\tan\theta = 0$$

Collecting like terms gives the following final *Quadratic Equation* for R:

$$R(D\tan^3\theta + \zeta C\tan^2\theta - \zeta - B\tan\theta) = 0$$

Or,

Equation 35. ***The Cubic Restitution Equation.***

$$D\tan^3\theta + \zeta C\tan^2\theta - B\tan\theta - \zeta = 0$$

Then for,

$$D\tan^3\theta + \zeta(C\tan^2\theta) - (B\tan\theta) - \zeta = 0$$

Substituting in *respective coefficients* from the *Generalized Cubic Equation 32* given below determines the following equation:

$$\beta = B\tan\theta$$

$$\gamma = C\tan^2\theta$$

$$\delta = D\tan^3\theta$$

$$\delta + \zeta\gamma - \beta - \zeta = 0$$

Or,

Equation 36. *ζ Relationship to Generalized Cubic Equation Coefficients.*

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$

TRUE SCANS

SECTION 12. *CHARACTERISTIC CUBIC EQUATION THRUWAY SYSTEM.*

Characteristic Cubic Equation 31 is a very compact expression whose coefficients are <u>inextricably linked</u> to *associated transforms* outlined in the prior section of this treatise.

In a sense *Equation 31* may be likened, or viewed as a *crossroads* which interconnects a plethora of other *associated transforms* by means of a so-called ***Thruway System***.

The latter portions of this section elucidate upon these associations, disclosing all of the intricate details which serve to bond, or cement the thruway. Each circumstance represents its own particular *byway* which sometimes delineates other supporting equations, and other times expresses known relationships portrayed earlier.

Table 16 represents a map, or chart that defines the entire thruway system. Byways appear as either given Characteristic Cubic *Equation 31* coefficients, or calculated values based upon them in combination with the tan θ.

Users are free to travel the thruway system and move from one equation, or transform, to another. However, in order to move from one point to another, as itemized on the map, *special rules apply* which appear in *Table 17*. Of interest, the rules *may* be different when attempting to return from a certain destination point back to an original embarkation point.

The *Table 16* map affords the logic which has been applied to the rules of *Table 17*. Since the only route from any of points 2, 3, or 4 to either point 5 or point 6 goes thru point 1, any attempt to transform from *Equation 30, Equation 33,* or *Equation 34 Quadratic* based representations to either *Equation 32,* or *Equation 35 Cubic* based transformations must go through *Equation 31*. *Equation 36* does not merit a destination point of its own since it is just a reformatting of point 6.

Examples for *Table 17* rules are given below for travel between points 1 and 2.

Table 16. Characteristic Cubic Equation Thruway System.

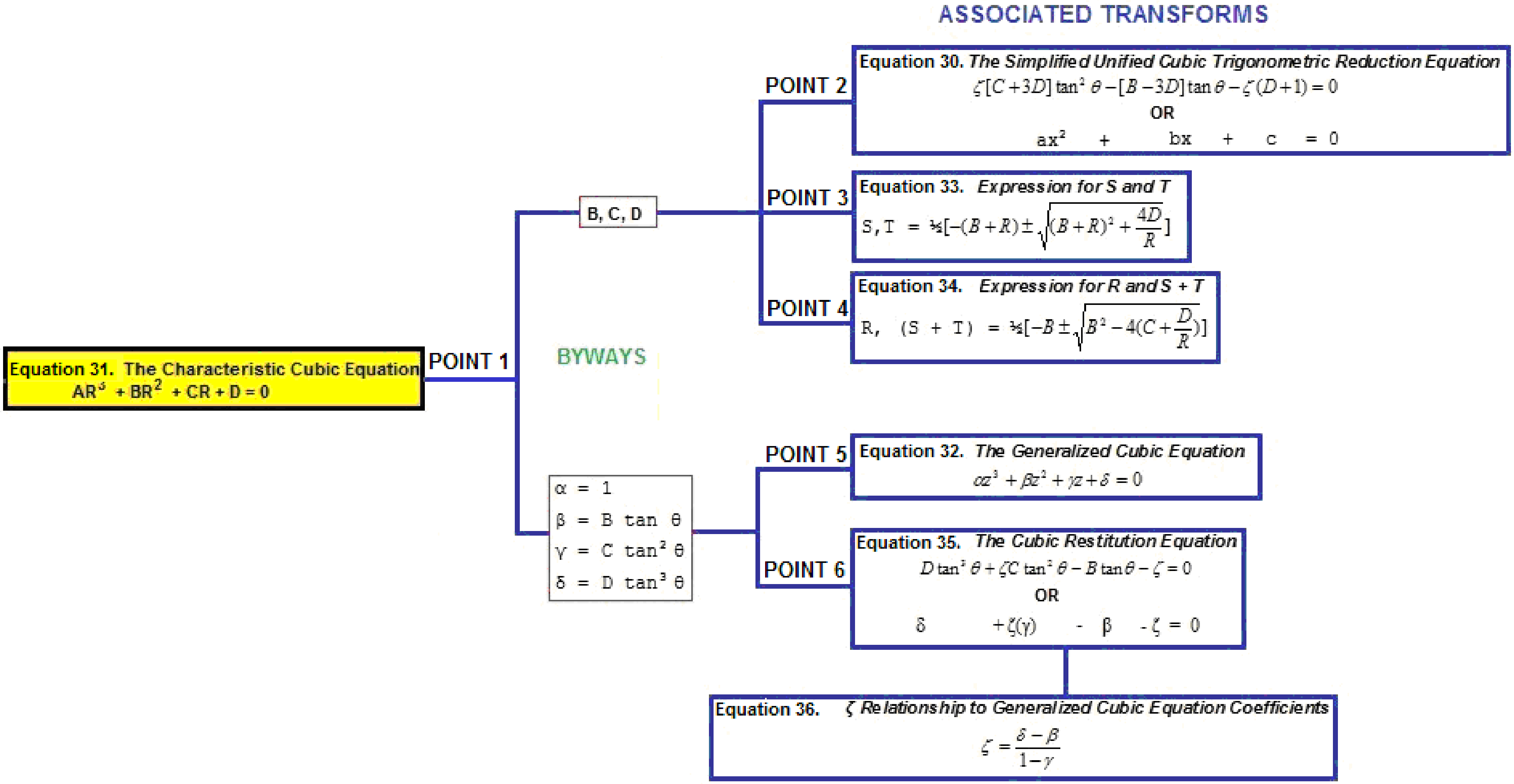


Table 17. Thruway Travel Rules.

Emb. Point	Dest. Point	Knowns or Givens	Unknowns	Rules
2	1	a, b, c	x, ζ, B, C, D	1) Calculate x = tan θ via *Quadratic Formula*; 2) Determine θ; 3) Determine 3θ; 4) Calculate ζ = tan (3θ); 5) Calculate D from c =- ζ(D+1); 6) Calculate B from b = (3D-B); 7) Calculate C from a = ζ(C+3D); 8) Write *Equation 31* for A = 1
1	2	B, C, D	x, ζ, a, b, c, R	1) Determine x^2 using *Equation 37*; 2) Calculate x = tan θ; 3) Determine θ; 4) Determine 3θ; 5) Calculate ζ = tan (3θ); 6) Calculate a = ζ(C+3D); 7) Calculate b = (3D-B)tan θ; 8) Calculate c =- ζ(D+1); 9) Write *Equation 30*
1	3 or 4	B, C, D	R, S, T	Requires Cubic Resolution – see next Section
3 or 4	1	R, S, T	B, C, D	1) Calculate B =-(R+S+T); 2) Calculate C = RS+RT+ST; 3) Calculate D = -RST; 4) Write *Equation 31* for A = 1
1	5	B, C, D	α, β, γ, δ, tan θ, R, z	1) Determine x^2 using *Equation 37*; 2) Calculate x = tan θ; 3) α = 1 4) Calculate β = B tan θ; 5) Calculate γ = C $\tan^2$ θ; 6) Calculate δ = D $\tan^3$ θ 7) Write *Equation 32*
1	6	B, C, D	α, β, γ, δ, tan θ, ζ, R, z	1) Determine x^2 using *Equation 37*; 2) Calculate x = tan θ; 3) Determine θ; 4) Determine 3θ; 5) Calculate ζ = tan (3θ); 6) α = 1 7) Calculate β = B tan θ; 8) Calculate γ =- C $\tan^2$ θ; 9) Calculate δ = D $\tan^3$ θ 10) Write *Equation 35*
5	1	α = 1, β,	B, C, D, ζ, tan θ,	1) Calculate ζ = tan (3θ) using *Equation 36*;

Emb. Point	Dest. Point	Knowns or Givens	Unknowns	Rules
		γ, δ	R, z	2) Determine 3θ; 3) Determine θ = 1/3 (3θ); 4) Calculate tan θ 7) Calculate B from β = B tan θ; 8) Calculate C from γ =- C $\tan^2$ θ; 9) Calculate D from δ = D $\tan^3$ θ 10) Write *Equation 31* for A = 1
6	1	α = 1, β, ζ (γ), δ, ζ	B, C, D, tan θ, R, z	1) From ζ = tan (3θ), determine 3θ; 2) Determine θ = 1/3 (3θ); 3) Calculate tan θ 4) Calculate B from β = B tan θ; 5) Determine γ = ζ (γ)/ ζ 6) Calculate C from γ =- C $\tan^2$ θ; 7) Calculate D from δ = D $\tan^3$ θ 8) Write *Equation 31* for A = 1

When traveling from Point 2 to Point 1, the *Characteristic Cubic Equation* [Ref. *Equation 31*] coefficients B, C and D can be determined from the coefficients a, b and c appearing in any given *Quadratic Equation*.

This may be achieved by applying the *Quadratic Formula* to the given coefficients a, b, and c in order to determine the first root of such given *Quadratic Equation* as follows:

$$x_1 = \tan\theta = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

From this root, ζ = tan (3θ) is easily established by first determining θ, then multiplying it by three, and lastly calculating its tangent.

Whereas *Equation 30* gives the following identities:

$$a = \zeta[C+3D]$$

$$b = 3D - B$$

$$c = -\zeta(D+1)$$

The unknowns B, C, and D may be determined as follows, where D needs to be determined before B, and C:

$$B = 3D - b$$

$$C = \frac{a}{\zeta} - 3D$$

$$D = -\frac{c+\zeta}{\zeta}$$

TRUE SCANS

With *Characteristic Cubic Equation* coefficients fully determined, an actual writing of the complete transform $AR^3 + BR^2 + CR + D = 0$ now becomes possible for A = 1.

This approach is demonstrated via example. With respect to the following given *Quadratic Equation*:

$$ax^2 + bx + c = 0$$

$$14x^2 + 16.2x - 35.625 = 0$$

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-16.2 \pm \sqrt{(16.2)^2 - 4(14)(-35.625)}}{2(14)}$$

$$= \frac{-16.2 \pm \sqrt{262.44 + 1995}}{28}$$

$$= \frac{-16.2 \pm \sqrt{2257.44}}{28}$$

$$= \frac{-16.2 \pm 47.51252466}{28}$$

$$= 1.118304452; -2.275447309$$

At,

$$x_1 = \tan\theta = 1.118304452$$

$$\theta = 48.19657147^o$$

$$3\theta = 144.5897144^o$$

$$\zeta = \tan(3\theta) = -0.710933225$$

$$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$

$$A = 1$$

$$D = -\frac{c + \zeta}{\zeta} = \frac{35.625 + 0.710933225}{-0.710933225} = -51.11019143$$

$$B = 3D - b = 3D - 16.2 = 3(-51.11019143) - 16.2 = -169.5305743$$

$$C = \frac{a}{\zeta} - 3D = \frac{14}{-0.710933225} + 3(51.11019143) = 133.6381482$$

Then, the associated *Characteristic Cubic Reduction Equation* is:

$$AR^3 + BR^2 + CR + D = 0$$

$$R^3 - 169.5305743R^2 + 133.6381482R - 51.11019143 = 0$$

Conversely, when traveling from Point 1 to Point 2, determining *the Quadratic Equation* which is associated with any given *Characteristic Cubic Equation 31* is considerably more difficult.

However, this too may be accomplished via proper interpretation of *Equation 31* coefficients B, C, and D in order to determine tan θ, and subsequent substitution of these results, along with a calculated value for ζ into the following three equations:

$$a = \zeta[C + 3D]$$
$$b = -B + 3D$$
$$c = \zeta(-D - 1)$$

Where,

$$ax^2 + bx + c = 0$$
$$\zeta[C + 3D]x^2 + (3D - B)x - \zeta(D + 1) = 0$$

Then,

$$\zeta[(C + 3D)x^2 - (D + 1)] = (B - 3D)x$$
$$\zeta = \frac{(B - 3D)x}{(C + 3D)x^2 - (D + 1)}$$
$$= \frac{\tan\theta(3 - \tan^2\theta)}{1 - 3\tan^2\theta}$$
$$= \frac{x(3 - x^2)}{1 - 3x^2}$$
$$\frac{(B - 3D)}{(C + 3D)x^2 - (D + 1)} = \frac{(3 - x^2)}{1 - 3x^2}$$

Cross-multiplying yields:

$$[(C + 3D)x^2 - (D + 1)](3 - x^2) = (B - 3D)(1 - 3x^2)$$

Letting x^2 = v gives:

$$[(C + 3D)v - (D + 1)](3 - v) = (B - 3D)(1 - 3v)$$

Or,

$$[(C + 3D)v - (D + 1)](3 - v) + (3D - B)(1 - 3v) = 0$$

Then,

$$3(C + 3D)v - 3(D + 1) - (C + 3D)v^2 + (D + 1)v + 3D - B - 3(3D - B)v = 0$$
$$-(C + 3D)v^2 + [(3C + 9D) + (D + 1) - (9D - 3B)]v - (3 + B) = 0$$
$$(C + 3D)v^2 - [(3C + 9D) + (D + 1) - (9D - 3B)]v + (3 + B) = 0$$
$$(C + 3D)v^2 - [3(B + C) + (D + 1)]v + (3 + B) = 0$$

Where,

$$v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{[3(B+C)+(D+1)] \pm \sqrt{[3(B+C)+(D+1)]^2 - 4(C+3D)(3+B)}}{2(C+3D)}$$

$$= \frac{[3(B+C)+(D+1)] \pm \sqrt{9(B^2+2BC+C^2)+6(B+C)(D+1)+(D+1)^2-[12C+4BC+36D+12BD]}}{2(C+3D)}$$

$$= \frac{[3(B+C)+(D+1)] \pm \sqrt{9B^2+18BC+9C^2)+6BD+6CD+6B+6C+(D^2+2D+1)-[12C+4BC+36D+12BD]}}{2(C+3D)}$$

Equation 37. ***Determination of tan²θ from Equation 31 Coefficients.***

$$\tan^2\theta = \frac{[3(B+C)+(D+1)] \pm \sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}}{2C+6D}$$

Applying *Equation 37* with respect to the *Characteristic Cubic Equation* determined in the above example gives:

$$AR^3 + BR^2 + CR + D = 0$$

$$R^3 - 169.5305743R^2 + 133.6381482R - 51.11019143 = 0$$

Then,

$A = 1$
$B = -169.5305743$
$C = 133.6381482$
$D = -51.11019143$

$$\tan^2\theta = \frac{[3(B+C)+(D+1)] \pm \sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}}{2C+6D} \quad \textit{[Ref. Equation 37]}$$

$$= \frac{-157.7874697 \pm \sqrt{419397.9325+2612.251668-317180.528-51988.44063+-40981.62802+1-1017.183446-801.8288892+1737.746509}}{-39.38485218}$$

$$= \frac{-157.7874697 \pm \sqrt{11,779.3215}}{-39.38485218}$$

$$= \frac{-157.7874697 \pm 108.5325827}{-39.38485218}$$

$$= 1.250604848$$

Then,

$$x = \tan\theta = \sqrt{1.250604848}$$
$$= 1.118304452$$

$$\theta = 53.55174607^o$$
$$3\theta = 160.6552382^o$$

$$\tan(3\theta) = \zeta = -0.710933225$$

Or,

$$a = \zeta[C + 3D] = -0.710933225[133.63814826 + 3(-51.11019143)] = 14$$
$$b = -B + 3D = -(-169.530574) + 3(-51.11019143) = 16.2$$
$$c = \zeta(-D - 1) = -0.710933225(51.11019143 - 1) = -35.625$$

Hence,

$$\mathrm{ax}^2 + \mathrm{bx} + \mathrm{c} = 0$$
$$14\mathrm{x}^2 + 16.2\mathrm{x} - 35.625 = 0$$ Q.E.D.

When *Characteristic Cubic Equation 31* coefficients combine to form a negative summation for the terms contained under the radical expressed in *Equation 37*, as indicated below, only *imaginary values* for the tan θ can result.

$$9(B^2 + C^2) + D^2 + 14BC - 6BD + 6CD + 1 + 6B - 6C - 34D < 0$$

This applies when large values of D appear in proportion to B and C. Once D becomes greater than 34, the D^2 term tends to dominate, thereby making the summation of the expression turn positive.

TRUE SCANS

SECTION 13. CUBIC RESOLUTION.

A Cubic Equation may be resolved easily once its ***coefficient structure*** becomes interpreted.

This consists of determining which of the following five categories it fits into; thereby resolving it by the approach specified:

CATEGORY 1: When R, S, or T **Sub-element** equals unity. [***Resolution Approach*** -- *Section 13.1*];

CATEGORY 2: When $\beta^2 = 3\alpha\gamma$ [***Resolution Approach*** -- *Section 13.2*];

CATEGORY 3: When R, S, or T **Sub-element** is <u>not</u> equal to unity [***Resolution Approach*** -- *Section 13.3*]; and

CATEGORY 4: When $\alpha = \gamma = 1$, $\beta = -2a$, $z = b = \beta^2/4$, and $\delta = c$ satisfies the relationship $b^3 + \beta b^2 + \gamma b + \delta = 0$ [***Resolution Approach*** -- *Section 13.4*].

CATEGORY 5: When either $\beta = \gamma = 0$ or $\gamma = -\beta z_R$ [***Resolution Approach*** -- *Section 13.5*].

13.1. Cubic Resolution when R, S, or T Sub-element Equals Unity.

Generalized Cubic Equations comprised of an R, S, or T term which equals unity are easily distinguished because the sums of the coefficients of their associated *Characteristic Cubic Equations* always equal zero. This is demonstrated as follows, where:

$$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$

$$AS^3 + BS^2 + CS + D = 0$$
$$AT^3 + BT^2 + CT + D = 0$$
$$A(1)^3 + B(1)^2 + C(1) + D = 0$$

$$A + B + C + D = 0$$

Now setting A, and R in this case, equal to unity enables a simple determination of remaining and S and T terms as follows:

Where,

$$B = -(R + S + T)$$
$$= -(1 + S + T)$$

$$D = -RST$$
$$= -(1)ST$$
$$= -ST$$

TRUE SCANS

Since B and D are identified in the given *Characteristic Cubic Equation*:

$$S + T = -(B+1)$$

$$-\frac{D}{T} + T = -(B+1)$$

$$-D + T^2 = -(B+1)T$$

$$T^2 + (B+1)T = D$$

$$T^2 + (B+1)T + (\frac{B+1}{2})^2 = D(\frac{4}{4}) + (\frac{B+1}{2})^2$$

$$(T + \frac{B+1}{2})^2 = D(\frac{4}{4}) + (\frac{B+1}{2})^2$$

$$T + \frac{B+1}{2} = \frac{1}{2}\sqrt{4D + (B+1)^2}$$

$$T = \frac{1}{2}[-(B+1) \pm \sqrt{4D + (B+1)^2}]$$

$$S = -(B+1+T)$$

$$= -(B+1) - \frac{1}{2}[-(B+1) \pm \sqrt{4D + (B+1)^2}]$$

$$= -\frac{1}{2}(2)(B+1) - \frac{1}{2}[-(B+1) \pm \sqrt{4D + (B+1)^2}]$$

$$= -\frac{1}{2}[B+1 \pm \sqrt{4D + (B+1)^2}]$$

The related *Generalized Cubic Equation* is established by equating *Equation 3* to *Equation 36* as follows:

$$\tan^3\theta = 3\tan\theta - \tan(3\theta)(1 - 3\tan^2\theta)$$ *[Ref. Equation 3]*

$$\tan(3\theta)(1 - 3\tan^2\theta) = 3\tan\theta - \tan^3\theta$$

$$\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

$$\tan(3\theta) = \frac{D\tan^3\theta - B\tan\theta}{1 - C\tan^2\theta}$$

Hence,

$$\frac{\tan\theta(3 - \tan^2\theta)}{1 - 3\tan^2\theta} = \frac{\tan\theta(D\tan^2\theta - B)}{1 - C\tan^2\theta}$$

$$\frac{3 - \tan^2\theta}{1 - 3\tan^2\theta} = \frac{D\tan^2\theta - B}{1 - C\tan^2\theta}$$

TRUE SCANS

By cross-multiplying, the following relationship between tan θ and the coefficients of any given *Characteristic Cubic Equation* are determined:

$$(3-\tan^2\theta)(1-C\tan^2\theta)=(1-3\tan^2\theta)(D\tan^2\theta-B)$$

$$3-\tan^2\theta-3C\tan^2\theta+C\tan^4\theta=D\tan^2\theta-B-3D\tan^4\theta+3B\tan^2\theta$$

$$3-(1+3C)\tan^2\theta+C\tan^4\theta=-B+(3B+D)\tan^2\theta-3D\tan^4\theta$$

$$(C+3D)\tan^4\theta-(1+3B+3C+D)\tan^2\theta+(3+B)=0$$

$$\tan^4\theta-(\frac{1+3B+3C+D}{C+3D})\tan^2\theta+\frac{3+B}{C+3D}=0$$

$$\tan^4\theta+\frac{b}{a}\tan^2\theta+\frac{c}{a}=0$$

By *twice* applying the *Quadratic Formula*, the final relationship is obtained as follows:

$$\tan^2\theta=\frac{1}{2}[-\frac{b}{a}\pm\sqrt{(\frac{b}{a})^2-4(1)\frac{c}{a}}]$$

$$=\frac{1}{2}[(\frac{1+3B+3C+D}{C+3D})\pm\sqrt{(\frac{1+3B+3C+D}{C+3D})^2-4(\frac{3+B}{C+3D})}]$$

$$\tan\theta=\frac{\sqrt{2}}{2}\sqrt{(\frac{1+3B+3C+D}{C+3D})\pm\sqrt{(\frac{1+3B+3C+D}{C+3D})^2-4(\frac{3+B}{C+3D})}}$$

Now, the related *Generalized Cubic Equation* may be developed by applying the following formula:

$$\alpha z^3+\beta z^2+\gamma z+\delta=0$$ [Ref. Equation 32]

$$z^3+(B\tan\theta)z^2+(C\tan^2\theta)z+D\tan^3\theta=0$$

A simple *verification* for the *Equation 3 Cubic* is shown for the case when ζ equals $\sqrt{3}$:

$$\tan^3\theta=3\tan\theta-\tan(3\theta)(1-3\tan^2\theta)$$ [Ref. Equation 3]

Letting 'z' represent the tan θ, and 'ζ' stand for the tan(3θ) gives:

$$z^3=3z-\zeta(1-3z^2)$$

$$z^3-3\zeta z^2+-3z+\zeta=0$$

$$z^3+(B\tan\theta)z^2+(C\tan^2\theta)z+D\tan^3\theta=0$$

$$B=\frac{-3\zeta}{\tan\theta}=\frac{-3\sqrt{3}}{\tan 20^o}=\frac{-3\sqrt{3}}{0.363970234}=-14.27631145$$

$$C=\frac{-3}{\tan^2\theta}=\frac{-3}{\tan^2 20^o}=\frac{-3}{(0.363970234)^2}=-22.64589651$$

$$D=\frac{\zeta}{\tan^3\theta}=\frac{\sqrt{3}}{\tan^3 20^o}=\frac{\sqrt{3}}{(0.363970234)^3}=35.92220796$$

Where,

$$\tan^2\theta = \frac{1}{2}[(\frac{1+3B+3C+D}{C+3D}) \pm \sqrt{(\frac{1+3B+3C+D}{C+3D})^2 - 4(\frac{3+B}{C+3D})}]$$

$$= \frac{1}{2}[(\frac{-73.84441592}{85.12072737}) \pm \sqrt{(\frac{1+3B+3C+D}{C+3D})^2 - 4(\frac{3+B}{C+3D})}]$$

$$= \frac{1}{2}[-0.867525668 \pm \sqrt{(0.867525668)^2 + 4(\frac{11.27631145}{85.12072737})}]$$

$$= \frac{1}{2}[-0.867525668 \pm 1.132474331]$$

$$= 0.132474331; -1$$

$$\tan\theta = \sqrt{0.132474331}$$
$$= 0.363970234$$
$$= \tan 20^o$$

Q.E.D.

13.2. *Cubic Resolution* when $\beta^2 = 3\alpha\gamma$.

Cubic resolution via *coefficient manipulation* is possible under the condition when $\beta^2 = 3\alpha\gamma$ as follows:

Where the *cubic binomial expansion* states:

$$(A+B)^3 = (A)^3 + 3(A)^2B + 3(A)B^2 + B^3$$

Then, it follows that:

$$(3\alpha z + \beta)^3 = 27\alpha^3 z^3 + 27\alpha^2 z^2\beta + 9\alpha z\beta^2 + \beta^3$$

For the given condition when $\beta^2 = 3\alpha\gamma$,

$$(3\alpha z + \beta)^3 = 27\alpha^3 z^3 + 27\alpha^2\beta z^2 + 27\alpha^2\gamma z + \beta^3$$

Now, with respect to the *Generalized Cubic Equation 32*:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$ *[Ref. Equation 32]*

Multiplying thru by $27\alpha^2$ yields,

$$27\alpha^3 z^3 + 27\alpha^2\beta z^2 + 27\alpha^2\gamma z + 27\alpha^2\delta = 0$$

Or,

$$27\alpha^3 z^3 + 27\alpha^2\beta z^2 + 27\alpha^2\gamma z = -27\alpha^2\delta$$

Substituting this expression into the above equation gives,

$$(3\alpha z + \beta)^3 = \beta^3 - 27\alpha^2\delta$$

$$3\alpha z + \beta = \sqrt[3]{\beta^3 - 27\alpha^2\delta}$$

Or,

TRUE SCANS

$$z = \frac{-\beta + \sqrt[3]{\beta^3 - 27\alpha^2\delta}}{3\alpha}$$

As an example, consider the following *specific Generalized Cubic Equation (Ref. Equation 32)*, fully designated except for its second term:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$

$$4z^3 + \beta z^2 + 6z + 16 = 0$$

For,

$$\beta^2 = 3\alpha\gamma$$
$$= 3(4)(6)$$
$$= 72$$

$$\beta = \pm\sqrt{72}$$

Then, the following *specific Generalized Cubic Equation (Ref. Equation 32)* meets the criterion that $\beta^2 = 3\alpha\gamma$:

$$4z^3 \pm \sqrt{72}z^2 + 6z + 16 = 0$$

Where,

$$z = \frac{-\beta + \sqrt[3]{\beta^3 - 27\alpha^2\delta}}{3\alpha}$$

$$= \frac{\mp\sqrt{72} + \sqrt[3]{\pm\sqrt{72}^3 - 27(4)^2(16)}}{3(4)}$$

$$= \frac{-\sqrt{72} + \sqrt[3]{-6301.059741}}{12}; \frac{+\sqrt{72} + \sqrt[3]{-7522.940259}}{12}$$

$$= \frac{-8.485281374 - 18.47018303}{12}; \frac{+8.485281374 - 19.59427524}{12}$$

$$= -2.2462887; -0.925749489$$

$$4z^3 + \sqrt{72}z^2 + 6z + 16 = 0$$

$$4(-2.2462887)^3 + \sqrt{72}(-2.2462887)^2 + 6(-2.2462887) + 16 = 0$$

$$-45.33741023 + 42.81514243 - 13.4777622 + 16 = 0$$

$$-16 + 16 = 0$$

$$0 = 0$$

$$4z^3 - \sqrt{72}z^2 + 6z + 16 = 0$$

$$4(-0.925749489)^3 - \sqrt{72}(-0.925749489)^2 + 6(-0.925749489) + 16 = 0$$

TRUE SCANS

$$-3.173514115 - 7.271988949 - 5.554496934 + 16 = 0$$

$$-16 + 16 = 0$$

$$0 = 0$$

Moreover, for conditions when $\alpha = 1$

$$\beta^2 = 3\alpha\gamma$$

$$\beta^2 = 3(1)\gamma$$

$$\frac{\beta^2}{3} = \gamma$$

Substituting into *Equation 32 renders:*

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 + \beta z^2 + \frac{\beta^2}{3} z + \delta = 0$$

This analysis demonstrates that the cubic variable 'z' for the particular *Generalized Cubic Equation* stipulated above may be determined by a very simple *manipulation of its coefficients*.

As an example of this, consider the following given *Generalized Cubic Equation*:

$$z^3 + 63z^2 + 1323z + 21.2 = 0$$

Where,

$$\gamma = \frac{\beta^2}{3} = \frac{63(63)}{3} = \frac{63(21)(3)}{3}$$

$$= 63(21)$$

$$= 1323$$

$$\alpha = 1$$

Then,

$$z_1 = \frac{-\beta + \sqrt[3]{\beta^3 - 27\delta}}{3}$$

$$= \frac{-63 + \sqrt[3]{63^3 - 27(21.2)}}{3}$$

$$= \frac{-63 + \sqrt[3]{249474.6}}{3}$$

$$= \frac{-63 + 62.95189071}{3}$$

$$= -0.01603643$$

For the *particular case* when $\beta = 3$, the specific *Generalized Cubic Equation* cited above further simplifies into the following form:

$$z^3 + \beta z^2 + \frac{\beta^2}{3} z + \delta = 0$$

$$z^3 + 3z^2 + \frac{3^2}{3} z + \delta = 0$$

$$z^3 + 3z^2 + 3z + \delta = 0$$

This equation may be deemed the ***Fundamental Symmetric Cubic Equation*** since it constitutes $(z+1)^3 = 0$ when δ equals unity.

Below, *Equation* 36 is incorporated into this result as follows:

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

At,

$$\gamma = \frac{\beta^2}{3}$$

$$\zeta = \frac{\delta - \beta}{1 - \frac{\beta^2}{3}}$$

$$= \frac{3(\delta - \beta)}{3 - \beta^2}$$

Cross multiplying yields:

$$\zeta(3 - \beta^2) = 3(\delta - \beta)$$

$$\frac{\zeta(3 - \beta^2) + 3\beta}{3} = \delta$$

Then,

$$z^3 + \beta z^2 + \frac{\beta^2}{3} z + \delta = 0$$

$$z^3 + \beta z^2 + \frac{\beta^2}{3} z + \frac{\zeta(3 - \beta^2) + 3\beta}{3} = 0$$

For $\beta = 3$

$$z^3 + 3z^2 + 3z + (3 - 2\zeta) = 0$$

TRUE SCANS

Then,

$$z^3+3z^2+3z+1+(2-2\zeta)=0$$
$$(z+1)^3+2(1-\zeta)=0$$
$$(z+1)^3=2(\zeta-1)$$
$$z+1=\sqrt[3]{2(\zeta-1)}$$

For the particular case when ζ = √3:

$$z+1=\sqrt[3]{1.464101615}$$
$$z+1=1.135508545$$
$$z=0.135508545$$

Check:

$$z^3+3z^2+3z+(3-2\zeta)=0$$
$$(0.135508545)^3+3(0.135508545)^2+3(0.135508545)+(3-2\zeta)=0$$
$$0.002488284+0.055087696+0.406525632+(3-2\sqrt{3})=0$$
$$3.464101613-2\sqrt{3}=0$$
$$1.732050807-\sqrt{3}=0$$
$$0=0$$

This solution is corroborated again using the above formula as follows:

$$z_1=\frac{-\beta+\sqrt[3]{\beta^3-27\delta}}{3}$$
$$=\frac{-3+\sqrt[3]{3^3-27(3-2\zeta)}}{3}$$
$$=\frac{-3+\sqrt[3]{27-27(3-2\zeta)}}{3}$$
$$=\frac{-3+3\sqrt[3]{1-(3-2\zeta)}}{3}$$
$$=-1+1\sqrt[3]{1-(3-2\zeta)}$$
$$=-1+\sqrt[3]{-2(1-\zeta)}$$
$$=-1+\sqrt[3]{1.464101615}$$
$$=-1+1.135508545$$
$$=0.135508545$$

TRUE SCANS

13.3. The Cubic Resolution Transform.

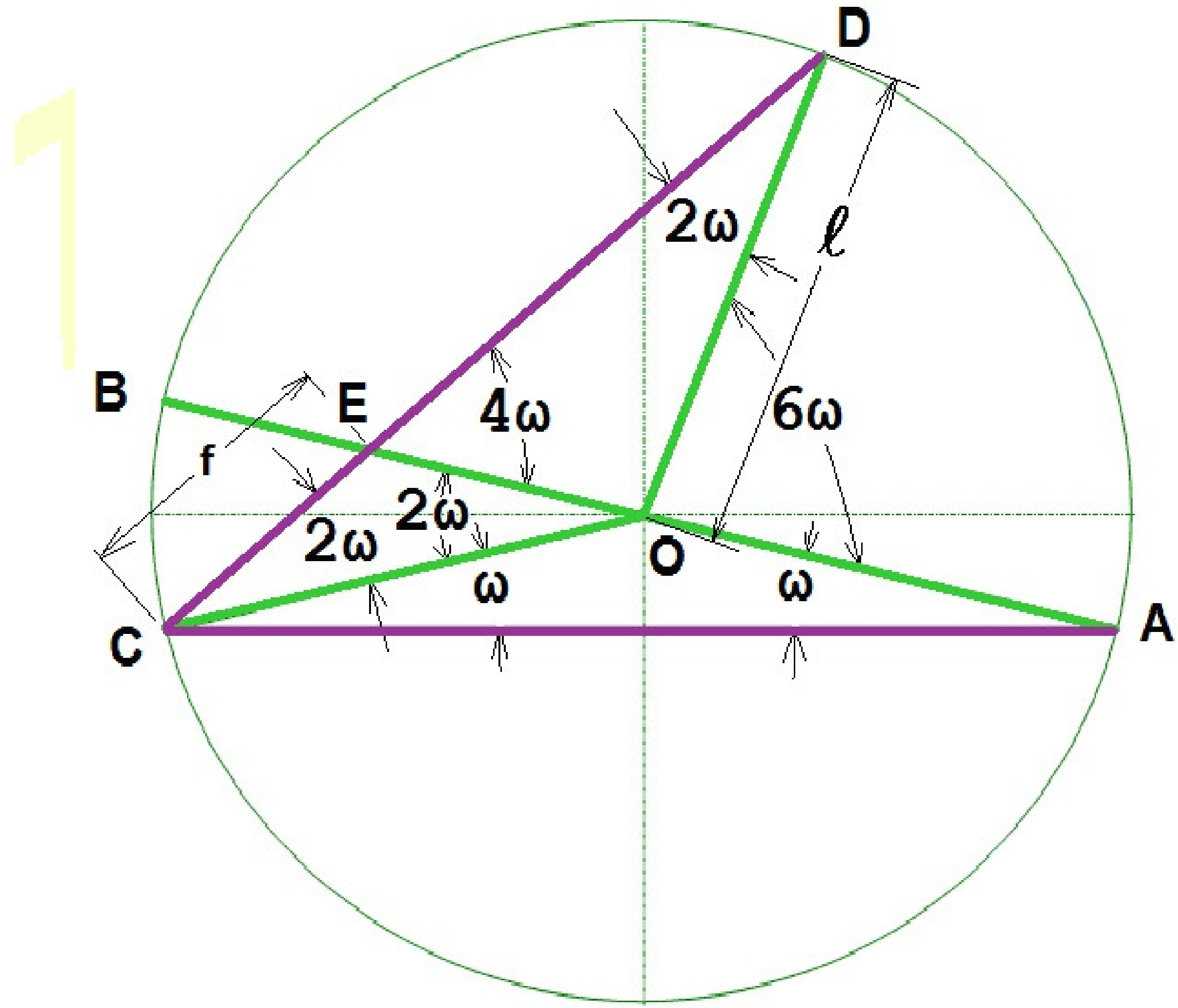
The new ***Cubic Resolution Transform (CRT)*** solves ***all*** given *Cubic Equations* ***directly***, regardless of what format *they* may assume.

Hence, such equations do not require that they first undergo a *transformation process* in order to remove any second order terms which they might contain.

13.3.1. Derivation.

The derivation of CRT is based upon the geometry afforded in *Figure 11*.

Figure 11. The Cubic Resolution Transform Geometry.



Derivation

ACTION	PROOF
1. In figure 11, draw circle with center at "O" and radius $\overline{OA}$.	1. Construction
2. Extend radius $\overline{OA}$ to Point B residing on circumference of circle.	2. Construction
3. Draw chord $\overline{AC}$ at any selected angle ω, where point C lies on the circumference of the circle.	3. Construction
4. Select the abscissa (x-axis) parallel to chord ☐	4. Construction
5. Draw radius $\overline{OC}$	5. Construction
6. Then $\angle OCA = \angle OAC = \omega$	6. The angles residing opposite equal sides of a triangle are equal
7. Draw chord $\overline{CD}$ to make an angle of "2ω" with radius $\overline{OC}$ where point D resides on the circumference of circle.	7. Construction
8. Draw radius $\overline{OD}$	8. Construction
9. Then $\angle ACD = \angle ACO + \angle OCD = 3\omega$	9. The whole is equal to the sum of its parts
10. $\angle AOD = 6\omega$	10. The central angle of a circle is equal to twice its exterior angle
11. $\angle OCA + \angle OAC + \angle COA = 180^o$	11. The sum of the angles of a triangle equal 180 degrees
12. $\omega + \omega + \angle COA = 180^o$	12. Substitution of 6
13. $2\omega = 180^o - \angle COA$	13. Subtracting ∠COA from both sides
14. $\angle COB + \angle COA = 180^o$	14. There are 180 degrees in a straight line
15. $\angle COB = 180^o - \angle COA$	15. Subtracting ∠COA *from both sides*

TRUE SCANS

ACTION	PROOF
16. Therefore $\overline{CE} = \overline{EO}$	16. The sides residing opposite equal angles of a triangle are equal
17. Designate $\overline{CE}$ as f $\overline{OA}$ as ℓ $\overline{ED}$ as v cos 6ω as ψ	17. Assignment
18. Then, $(\ell + f)(\ell - f) = v f$	18. The means equal the extremes (ref. similar triangles BDE and CEA)
19. $v^2 = f^2 + \ell^2 - 2f\ell\cos(180^o - 6\omega)$ $= f^2 + \ell^2 + 2f\ell\psi$	19. Law of Cosines[1]
20. $f^2 + \ell^2 + 2f\ell\psi = \left[\frac{(\ell + f)(\ell - f)}{f}\right]^2$	20. Substitution of line 18 into line 19
21. $f^4 + f^2\ell^2 + 2f^3\ell\psi = f^4 - 2f^2\ell^2 + \ell^4$	21. Cross multiplication and squaring of terms
22. $f^3(2\ell\psi) + 3f^2\ell^2 - \ell^4 = 0$	22. Collection of terms
23. $f^3 + \left(\frac{3l}{2\psi}\right)f^2 - \left(\frac{\ell^3}{2\psi}\right) = 0$	23. Division by common denominator and cancellation of terms

TRUE SCANS

1. CRC Standard Mathematical Tables Twelfth Edition; The Chemical Rubber Co. Cleveland, OH; January 1964; page 410.

13.3.2. Proof.

From *Equation 1*:

$$\cos^3\phi - \frac{3}{4}\cos\phi - \frac{\cos 3\phi}{4} = 0$$

Letting

$$\phi = 2\omega$$

Then

$$\cos^3(2\omega) - \frac{3}{4}\cos(2\omega) - \frac{\cos(6\omega)}{4} = 0$$

Next, substituting ψ for $\cos 6\omega$ gives:

$$\cos^3(2\omega) - \frac{3}{4}\cos(2\omega) - \frac{\psi}{4} = 0$$

Then, multiplying each term by $-\frac{\ell^3}{2\psi\cos^3(2\omega)}$ yields:

$$-\frac{\ell^3}{2\psi\cos^3(2\omega)}[\cos^3(2\omega)] + \frac{\ell^3}{2\psi\cos^3(2\omega)}[\frac{3}{4}\cos(2\omega)] + \frac{\ell^3}{2\psi\cos^3(2\omega)}(\frac{\psi}{4}) = 0$$

$$-\frac{\ell^3}{2\psi} + \frac{3\ell^3}{8\psi\cos^2(2\omega)} + \frac{\ell^3}{8\cos^3(2\omega)} = 0$$

$$\frac{\ell^3}{8\cos^3(2\omega)} + \frac{3\ell}{2\psi}\left(\frac{\ell^2}{4\cos^2(2\omega)}\right) - \frac{\ell^3}{2\psi} = 0$$

From triangle CEO of *Figure 11*, it is readily apparent that $f = \frac{\ell/2}{\cos(2\omega)}$

Hence,

$$f^3 = \frac{\ell^3}{8\cos^3(2\omega)}; \quad f^2 = \frac{\ell^2}{4\cos^2(2\omega)},$$ such that:

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right)f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0$$

The minus signs in the above term account for negative values of ψ with ℓ always being positive.

13.3.3. Assumed Forms.

Below, the CRT is transformed into various forms of the *Generalized Cubic Equation (Ref. Equation 32).* For some of these transforms, it is demonstrated that ψ, or cos (6ω), is ***coefficient driven***; that is, it is *fully determinable* by *coefficient manipulation.*

Equation 38*. Cubic Resolution Transform (CRT).*

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right) f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0$$

When $f = z$, the above CRT may be depicted by a *Generalized Cubic Equation* whose *third term coefficient* is equal to zero as follows:

$$z^3 + \sigma z^2 + \nu = 0$$

The CRT assumes the following *intermediate form* (obtained by letting $f = u + V$):

Equation 39*. CRT Intermediate Form.*

$$u^3 + \left(3V + \frac{3\ell}{2\psi}\right)u^2 + \left(3V^2 + \frac{3V\ell}{\psi}\right)u + \frac{3V^2\ell - \ell^3}{2\psi} + V^3 = 0$$

It assumes a *Transformed Intermediate Form* (when $V = -\ell/\psi$):

Equation 40*. CRT Transformed Intermediate Form.*

$$u^3 - \left(\frac{3\ell}{2\psi}\right)u^2 + \frac{\ell^3}{2\psi^3}\left(1 - \psi^2\right) = 0$$

Rearrangement of *Equation 38* terms permits resolution of *Cubic Equations* devoid of *second order terms* as follows:

Equation 41*. CRT Rearranged Form.*

$$\ell^3 - \left(3f^2\right)\ell - 2\psi\left(f^3\right) = 0$$

For $\ell = z$, the *CRT Rearranged Form* is depicted by a *Generalized Cubic Equation* whose *second term coefficient* is equal to zero as follows:

$$z^3 + \gamma z + \delta = 0$$

Comparing like coefficients renders:

$$\gamma = -3f^2$$

$$\delta = -2\psi f^3$$

Or,

$$f = \sqrt{-\frac{\gamma}{3}}$$

TRUE SCANS

$$\psi = \cos(6\varpi) = \frac{\delta}{-2f^3} = -\frac{\delta}{2(-\frac{\gamma}{3})^{\frac{3}{2}}}$$

However, since:

$$\cos(2\varpi) = \frac{\ell/2}{f} = \frac{z}{2f}$$ *[Ref. Figure 11]*

$$2f\cos(2\varpi) = z$$

Or,

$$z = 2\sqrt{-\frac{\gamma}{3}}\cos(2\varpi)$$

The relationship *directly above* is of use for determining a value for z represented in equations of the following *Generalized Cubic Equation* ***reduced form*** under conditions when the R, S, and T ***sub-element*** *terms* are not equal to unity *(Ref. Section 13.1)*; that is, $z \neq \tan\theta$ ($\tan\theta$ being determined from ζ as derived in *Equation 36*).

$$z^3 + \gamma z + \delta = 0$$

The *CRT Rearranged Transformed Form* is achieved when $\ell = u + V$:

Equation 42. ***CRT Rearranged Transformed Form.***

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0$$

This specifies the actual *Generalized Cubic Equation format* as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$ *[Ref. Equation 32]*

For $\alpha = 1$ and $z = u$, comparing like coefficients gives:

$$\beta = 3V$$

$$\gamma = 3(V^2 - f^2)$$

$$\delta = V^3 - 3f^2V - 2\psi f^3$$

Or,

$$V = \frac{\beta}{3}$$

$$f^2 = V^2 - \frac{\gamma}{3}$$

$$f = \sqrt{V^2 - \frac{\gamma}{3}}$$

$$= \sqrt{(\frac{\beta}{3})^2 - \frac{\gamma}{3}(\frac{3}{3})}$$

TRU SCANS

$$= \frac{1}{3}\sqrt{\beta^2 - 3\gamma}$$

$$\psi = \frac{V^3 - 3f^2V - \delta}{2f^3}$$

$$= \frac{V^3 - 3(V^2 - \frac{\gamma}{3})V - \delta}{2(\frac{1}{3})^3(\beta^2 - 3\gamma)^{\frac{3}{2}}}$$

$$= \frac{-2V^3 + \gamma V - \delta}{\frac{2}{27}(\beta^2 - 3\gamma)^{\frac{3}{2}}}$$

$$= \frac{-2(\frac{\beta}{3})^3 + \gamma(\frac{\beta}{3})(\frac{9}{9}) - \delta(\frac{27}{27})}{\frac{2}{27}(\beta^2 - 3\gamma)^{\frac{3}{2}}}$$

$$= \frac{9\gamma\beta - 2\beta^3 - 27\delta}{2(\beta^2 - 3\gamma)^{\frac{3}{2}}}$$

However, since:

$$\cos(2\varpi) = \frac{\ell/2}{f} \qquad \textit{[Ref. Figure 11]}$$

$$= \frac{\ell}{2f}$$

$$= \frac{z+V}{2f}$$

$$2f\cos(2\varpi) - V = z$$

Or,

$$z = -\frac{1}{3}[\beta - 2\sqrt{\beta^2 - 3\gamma}\cos(2\varpi)]$$

The relationship *directly above* is of use for determining a value for z represented in *Generalized Cubic Equations* under conditions when the R, S, and T ***sub-element*** *terms* are <u>not</u> equal to unity *(Ref. Section 13.1)*; that is, $z \neq \tan\theta$ ($\tan\theta$ being determined from ζ as derived in *Equation 36*).

The transformed intermediate form results when $V = f$:

Equation 43*. CRT Rearranged Transformed Intermediate Form.*

$$u^3 + (3f)u^2 - 2f^3(1+\psi) = 0$$

TRUE SCANS

13.3.4. Real Root Solution.

CRT equations expressed above all incorporate, or make reference to, a factor ψ which is equal to the cos (6ω). This sub-section pertains to the cases where:

$$-1 \le \cos(6\omega) \le +1 \qquad \textit{[Ref. Section 15.3]}$$

As example, *Characteristic Cubic Equation 31* is resolved below for the specific case when $R=2$, $S=3$ and $\zeta=13/9$.

First, T is calculated in order to assure that it produces a *real root*. Since imaginary roots occur in pairs, this is really a moot point.

$$\zeta = \tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^o$$

$$\theta = 18.43494882^o$$

$$\tan\theta = \frac{1}{3}$$

$$z_R = R\tan\theta = 2\tan\theta = 2(\frac{1}{3}) = \frac{2}{3} = \tan\theta_R$$

$$z_S = S\tan\theta = 3\tan\theta = 3(\frac{1}{3}) = 1 = \tan\theta_S$$

$$z_T = T\tan\theta = \tan\theta_T$$

$$\theta_R = \arctan(\frac{2}{3}) = 33.69006753^o$$

$$\theta_S = \arctan(1) = 45^o$$

$$\theta_R + \theta_S + \theta_T = 3\theta$$

$$\theta_T = 3\theta - (\theta_R + \theta_S)$$

$$= 55.30484647^o - (33.69006753^o + 45^o)$$

$$= -23.38522106^o$$

$$\tan\theta_T = -0.432432432 = T\tan\theta$$

$$\frac{-0.432432432}{1/3} = T$$

$$-1.297297297 =$$

Such that

$$B = -(R+S+T) = -3.702702702$$

$$C = R(S+T) + ST = -0.486486486$$

$$D = -RST = +7.783783783$$

TRUE SCANS

Now, the associated *Characteristic Cubic Equation is*

$$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$

$$R^3 - 3.702702702R^2 - 0.486486486R + 7.783783783 = 0$$

In order to resolve this above equation, first select a suitable format from *Section 13.3.3* which matches it, and then compare respective coefficients. This approach is demonstrated as follows:

Equation 42 is selected from the excerpt since it matches the above format,

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0$$

While viewing u terms to be synonymous with R values, respective coefficients are compared to return the following results,

$$V = \frac{-3.702702702}{3} = -1.234234234$$

$$f = \sqrt{V^2 + \frac{0.486486486}{3}} = 1.298266655$$

$$\psi = \frac{V^3 - 3f^2V - 7.783783783}{2f^3} = -0.78215113 = \cos(6\omega)$$

Secondly, resolve the unknown equation given above as follows:

$$6\omega = 141.457955^o, 218.542045^o, (141.457955^o + 360^o)$$

$$2\omega = \frac{141.457955^o}{3}; \frac{218.542045^o}{3}; \frac{501.457955^o}{3}$$

$$= 47.15265166^o; 72.84734834^o; 167.1526517^o$$

$$\cos(2\omega) = +0.680047415; +0.294918522; -0.974965937$$

$$2f\cos(2\omega) = (2f)[+0.680047415; +0.294918522; -0.974965937]$$

$$= (2.59653331)[+0.680047415; +0.294918522; -0.974965937]$$

$$= +1.765765765; +765765765; -2.531531531$$

$$= \ell$$

Now since,

$$u = \ell - V$$

$$= (+1.76575842; +765765765; -2.531531531) - (-1.234234234)$$

$$= +3; +2; -1.297297297$$

$$= R, S, T$$

TRUE SCANS

A second example resolution is presented below for the following equation:

$$R^3 - 4R^2 + 3R - \frac{1}{4} = 0$$

Substituting $R = u$ into the *above given equation* renders:

$$u^3 - 4u^2 + 3u - \frac{1}{4} = 0$$

Equation 42 is selected from the *Cubic Resolution Transform* excerpt since it matches the above format,

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0 \quad \textit{[Ref. Equation 42]}$$

Comparing respective coefficients returns the following results,

$$V = \frac{-4}{3} = -1.333333333$$

$$f = \sqrt{V^2 - \frac{3}{3}} = 0.881917103$$

$$\psi = \frac{V^3 - 3f^2V + 1/4}{2f^3} = 0.722182116 = \cos(6\omega)$$

Since $-1 \le \cos(6\omega) \le +1$ the *three roots* are *all real*. Hence,

$$6\omega = +43.76506489^o;-43.76506489^o;+(43.76506489^o + 360^o)$$
$$= +43.76506489^o;+316.2349351^o;+403.76506489^o$$

$$2\omega = \frac{+43.76506489^o}{3};\frac{+316.2349351^o}{3};\frac{+403.76506489^o}{3}$$
$$= +14.58835496^o;+105.411645^o;+134.588355^o$$

$$\cos(2\omega) = +0.967760382;-0.265752058;-0.702008323$$

$$2f\cos(2\omega) = (2f)(+0.967760382;-0.265752058;-0.702008323)$$
$$= (1.763834206)(+0.967760382;-0.265752058;-0.702008323)$$
$$= +1.706968865;-0.46874257;-1.238226293$$
$$= \ell$$

Now since,

$$u = \ell - V$$
$$= (+1.706968865;-0.46874257;-1.238226293) - (-1.333333333)$$
$$= +3.040302198;+0.864590763;+0.09510704$$
$$= R, S, T$$

TRUE SCANS

13.3.5. Imaginary Root Solution.

This sub-section pertains to the cases where $\cos(6\omega)$ is either less than minus one, greater than positive one (*Ref. Section 15.3*), or exists as a *complex variable* or *pure imaginary number (Ref. Section 20 -- Problem Number 30).*

The following given *Generalized Cubic Equation* is selected for resolution:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 + 3z^2 - 215.055544 = 0$$

In this case, its *first order term* is equal to zero.

Equation 38 is employed in order to resolve this given *Cubic Equation* as follows:

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right) f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0 \qquad \textit{[Ref. Equation 38]}$$

The variable 'f' appears as a representation of 'z' in the given *Cubic Equation*.

Secondly, a comparison between coefficients of like terms is carried out as follows:

$$3 = \frac{3\ell}{2\psi}$$

$$1 = \frac{\ell}{2\psi}$$

$$-215.055544 = -\frac{\ell^3}{2\psi}$$

$$= -\left(\frac{\ell}{2\psi}\right)\ell^2$$

$$= -(1)\ell^2$$

Then,

$$\ell^2 = 215.055544$$

$$\ell = \sqrt{215.055544}$$

$$= 14.66477221$$

From above,

$$\frac{\ell}{2\psi} = 1$$

$$\frac{\ell}{2} = \psi$$

$$7.332386106 = \cos(6\omega)$$

Since the $\cos(6\omega)$ lies *outside* of the −1 through +1 *real number regime*, this analysis adopts an *imaginary* numerical solution by making use of the following identity:

$$\cosh x = \cos(ix)^{FOOTNOTE1}$$

Then, by letting:

$$6\omega = ix$$

$$\cosh x = \cos(6\omega)$$

$$= 7.332386106$$

Or,

$$x = 2.680765435$$

$$ix = 2.680765435i = 6\omega$$

$$\frac{ix}{3} = \frac{2.680765435i}{3} = 2\omega$$

$$\frac{ix}{3} = 0.893588478i = 2\omega$$

$$\frac{x}{3} = 0.893588478$$

Accordingly,

$$\cosh\frac{x}{3} = \cos(\frac{ix}{3})$$

$$\cosh 0.89388478 = \cos(2\omega)$$

$$1.426534261 = \cos(2\omega)$$

Where,

$$\cos(2\omega) = \frac{\ell/2}{f}$$

$$1.426534261 = \frac{\ell}{2f}$$

$$f = \frac{14.66477221}{2(1.426534261)}$$

$$= 5.14$$

TRUE SCANS

1. CRC Standard Mathematical Tables Twelfth Edition; The Chemical Rubber Co. Cleveland, OH; January 1964; page 433.

Check:

$$z^3 + 3z^2 - 215.055544 = 0$$
$$(5.14)^3 + 3(5.14)^2 - 215.055544 = 0$$
$$135.796744 + 79.2588 - 215.055544 = 0$$
$$215.055544 - 215.055544 = 0$$
$$0 = 0$$

Q.E.D.

Now, the two remaining imaginary roots are determined as follows:

$$(z - 5.14)(z^2 + bz + c) = 0$$
$$z^3 + (b - 5.14)z^2 + (c - 5.14b)z - 5.14c = 0$$
$$z^3 + 3z^2 - 215.055544 = 0$$

Equating like terms yields:

$$-5.14c = -215.055544$$
$$c = 41.8396$$

$$b - 5.14 = 3$$
$$b = 8.14$$

Check,

$$c - 5.14b = 0$$
$$41.8396 - 5.14(8.14) = 0$$
$$41.8396 - 41.8396 = 0$$
$$0 = 0$$

Lastly, via *Quadratic Formula*

$$z_2, z_3 = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$$
$$z_2, z_3 = \frac{1}{2}[-8.14 \pm \sqrt{(8.14)^2 - 4(41.8396)}]$$
$$z_2, z_3 = \frac{1}{2}(-8.14 \pm \sqrt{-101.0988})$$

Check:

$$(z - 5.14)[z - \frac{1}{2}(-8.14 + \sqrt{-101.0988})][z - \frac{1}{2}(-8.14 - \sqrt{-101.0988})] = 0$$
$$[z - 5.14]\{z^2 + 8.14z + \frac{1}{4}[(-8.14)^2 - (-101.0988)]\} = 0$$
$$(z - 5.14)(z^2 + 8.14z + \frac{167.3584}{4}) = 0$$
$$(z - 5.14)(z^2 + 8.14z + 41.8396) = 0$$
$$z^3 + (8.14 - 5.14)z^2 + [41.8396 - 5.14(8.14)]z - 5.14(41.8396) = 0$$
$$z^3 + 3z^2 + (41.8396 - 41.8396)z - 215.055544 = 0$$
$$z^3 + 3z^2 - 215.055544 = 0$$

TRUE SCANS

13.3.6. CRT in Relation to Existing State-of-the-Art.

The CRT (Ref. *Equation 38*) resolves *Cubic Equations* ***directly***! It does so regardless of what format they may assume.

In stark contrast, basic present day resolutions cited below are limited in the sense that they can operate only upon *cubic formats* which are *devoid* of their second order terms:

- The *Trigonometric Solution of the Cubic Equation*[1]
- *Cardano's Method*[2]

Accordingly, such *presently accepted Cubic Resolution Techniques* require that *given Cubic Equations* first must undergo a *transformation process* before resolution can be made possible.

Now, the analysis presented below correlates such limited resolutions, as cited above, to a **broader** *overall CRT philosophy* which ***applies directly to all Cubic Equations***, *regardless* of format

In so doing, the CRT ascribes its **newly defined geometry** *(Ref. Figure 11)* to such well known, already existing *Trigonometric Solution of the Cubic Equation* posed in the first bullet above -- one whose sole purpose is to resolve the following *unknown equation* when known values for both "a" and "b" are supplied, or postulated:

$$x^3 + ax + b = 0$$ Footnote 3

Such resolution is achieved by letting $x = m\cos\theta$ Footnote 4.

Now, by substituting this expression into the *unknown equation*, the following equality is obtained:

$$x^3 + ax + b = 0$$
$$= m^3\cos^3\theta + am\cos\theta + b$$ Footnote 5

TRU SCANS

1. CRC Standard Mathematical Tables 23rd Edition; Samuel M. Selby - Editor in Chief; CRC Press, Inc, Cleveland Ohio; 1975; page 104.
2. On-line website address: *http://en.wikipedia.org/wiki/Cubic_function*
3. Op. cit.
4. Ibid.
5. Ibid.

This *unknown equation* ultimately becomes resolved by multiplying each of its derived terms by a factor of $4/m^3$, as follows:

$$4\cos^3\theta + \frac{4a}{m^2}\cos\theta + \frac{4b}{m^3} = 0$$

A particular value of "m" is calculated such that:

$$\frac{4a}{m^2} = -3$$

So, "m" must equal:

$$m = \sqrt{-\frac{4a}{3}}$$

When this occurs, the equation above reduces to:

$$4\cos^3\theta - 3\cos\theta + \frac{4b}{m^3} = 0$$

Multiplying each term in *Equation 1* by a factor of 4 produces the following result:

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta) \qquad \text{[Ref. Equation 1]}$$

$$4\cos^3\theta = 3\cos\theta + \cos(3\theta)$$

Or,

$$4\cos^3\theta - 3\cos\theta - \cos(3\theta) = 0$$

But,

$$4\cos^3\theta - 3\cos\theta + \frac{4b}{m^3} = 0$$

Or,

$$4\cos^3\theta - 3\cos\theta = \cos(3\theta)$$

$$4\cos^3\theta - 3\cos\theta = -\frac{4b}{m^3}$$

Then,

$$\cos(3\theta) = -\frac{4b}{m^3}$$

Hence, both "m" and the "cos (3θ) are determined by *manipulation of the given coefficients* "a" and "b" in the afforded *unknown equation*.

TRUE SCANS

Thereafter, the posed *unknown equation* may be resolved by first determining θ as follows:

Where,

$$3\theta = -\arccos\left(-\frac{4b}{m^3}\right)$$

Or,

$$\theta = \frac{1}{3}\arccos\left(-\frac{4b}{m^3}\right)$$

The value "x" is ascertained simply by multiplying the obtained value of "m" by that of "cos θ.

Secondly, both the CRT and its associated geometry, as alluded to above and afforded in *Figure 11*, are directly associated with *Equation 1* as follows:

From above:

$$4\cos^3\theta = 3\cos\theta + \cos(3\theta)$$

Or,

$$4\cos^3\theta - 3\cos\theta - \cos(3\theta) = 0$$

Now, substituting 2ω for θ yields:

$$4\cos^3(2\omega) - \cos(2\omega) - \cos(6\omega) = 0$$

Lastly, assigning ψ for 6ω renders:

$$4\cos^3(2\omega) - \cos(2\omega) - \psi = 0$$

The *CRT* represents a transform of this relationship as follows:

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right)f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0$$ *[Ref. Equation 38]*

Multiplying thru by $-\frac{\psi}{f^3}$ gives:

$$-\psi - \left(\frac{3\ell}{2f}\right) + \left(\frac{\ell^3}{2f^3}\right) = 0$$

Or,

$$\frac{\ell^3}{2f^3} - \frac{3\ell}{2f} - \psi = 0$$

And since,

$$f = \frac{\ell/2}{\cos(2\omega)}$$

TRUE SCANS

$$= \frac{\ell}{2\cos(2\omega)}$$

Or,

$$\ell = 2f\cos(2\omega)$$

Substituting this expression for ℓ above finally renders the following equality:

$$4\cos^3(2\omega) - 3\cos(2\omega) - \psi = 0 \qquad \text{Q.E.D.}$$

In conclusion, the *Trigonometric Solution of the Cubic Equation*[1] may be relegated to the geometry presented in *Figure 11* by ascertaining its relationship to the CRT as follows:

Since,

$$\frac{\ell^3}{2f^3} - \frac{3\ell}{2f} - \psi = 0$$

Multiplying each term by a factor of $2f^3$ establishes the *CRT Rearranged Form*:

$$\ell^3 - (3f^2)\ell - 2\psi(f^3) = 0 \qquad \textit{[Ref. Equation 41]}$$

Comparison of like terms with those expressed in the *unknown equation*, reiterated below, yields the following results when ℓ is permitted to assume the value of x:

$$\ell^3 - (3f^2)\ell - 2\psi(f^3) = 0$$

$$x^3 + ax + b = 0$$

$$-3f^2 = a$$

$$f = \sqrt{-\frac{a}{3}}$$

$$2f = 2\sqrt{-\frac{a}{3}}$$

$$= \sqrt{-\frac{4a}{3}}$$

$$= m$$

Or,

$$f = \frac{m}{2}$$

1. CRC Standard Mathematical Tables 23rd Edition; Samuel M. Selby - Editor in Chief; CRC Press, Inc, Cleveland Ohio; 1975; page 104.

$$-2\psi\left(f^3\right)=b$$

$$\psi=-\frac{b}{2f^3}$$

$$\psi=-\frac{b}{2(\frac{m}{2})^3}$$

$$\psi=-\frac{4b}{m^3}$$

$$=\cos(3\theta)$$

Therefore, when the value of "m/2" ascertained in the already existing *Trigonometric Solution of the Cubic Equation*[1] is permitted to portray length "f" shown in *Figure 11*, its associated ascertained value "3θ" may be represented by "6ω" in such figure.

Additionally, in order to successfully resolve an equation of the form $x^3+ax+b=0$ Footnote 2 by use of the *Trigonometric Solution of the Cubic Equation*[3], a thought to be unknown concept *first* should be applied which relegates a resulting mathematical analysis to be conducted in the *same* real, or imaginary regime as that afforded by any given equation.

This consists of selecting an equation format from the list specified below whose coefficients exhibit the same sign convention of such given equation:

- $4\cos^3\theta-3\cos\theta-\cos(3\theta)=0$ *[Ref. Equation 1]*
- $4\sin^3\theta-3\sin\theta+\sin(3\theta)=0$ *[Ref. Equation 2]*
- $4\sinh^3 x+3\sinh x-\sinh(3x)=0$ Footnote 4

For example, the equation $z^3+1.301288503z-0.521846994=0$ may be resolved as follows, where:

$$\alpha z^3+\beta z^2+\gamma z+\delta=0 \qquad \textit{[Ref. Equation 32]}$$

$$\alpha=1$$
$$\beta=0$$
$$\gamma=1.301288503$$
$$\delta=-0.521846994$$

1. CRC Standard Mathematical Tables 23rd Edition; Samuel M. Selby - Editor in Chief; CRC Press, Inc, Cleveland Ohio; 1975; page 104.
2. Ibid.
3. Ibid.
4. CRC Standard Mathematical Tables 12th Printing; Charles D. Hodgman, M.S. - Editor in Chief; The Chemical Rubber Company, Cleveland Ohio; 1964; page 432.

Since γ is positive and δ is negative, the third equation, $4\sinh^3 x + 3\sinh x - \sinh(3x) = 0$ *Footnote 1*, is selected from the list above because it matches the above given sign convention.

Then, letting,

$$z = m\sinh x$$

$$z^3 = m^3 \sinh^3 x$$

Via substitution:

$$z^3 + 1.301288503z - 0.521846994 = 0$$

$$z^3 + 0z^2 + \gamma z + \delta = 0$$

$$m^3 \sinh^3 x + \gamma m \sinh x + \delta = 0$$

Or,

$$4\sinh^3 x + 4\frac{\gamma}{m^2}\sinh x + 4\frac{\delta}{m^3} = 0$$ *Footnote 2*

Matching respective terms renders the following equalities:

$$4\sinh^3 x + 4\frac{\gamma}{m^2}\sinh x + 4\frac{\delta}{m^3} = 0$$

$$4\sinh^3 x + 3\sinh x - \sinh(3x) = 0$$

$$3 = 4\frac{\gamma}{m^2}$$

$$m^2 = \frac{4}{3}\gamma$$

$$m = 2\sqrt{\frac{\gamma}{3}}$$

$$m^3 = \frac{4}{3}\gamma(2\sqrt{\frac{\gamma}{3}})$$

$$= \frac{8}{3}\sqrt{\frac{\gamma^3}{3}}$$

$$\sinh(3x) = -4\frac{\delta}{m^3}$$

$$= -4\frac{\delta}{\frac{8}{3}\sqrt{\frac{\gamma^3}{3}}}$$

$$= -\frac{3\sqrt{3}\delta}{2\sqrt{\gamma^3}}$$

1. <u>CRC Standard Mathematical Tables</u> 12th Printing; Charles D. Hodgman, M.S. - Editor in Chief; The Chemical Rubber Company, Cleveland Ohio; 1964; page 432.
2. Ibid.

$$\sinh(3x) = +\frac{3\sqrt{3}(0.521846994)}{2\sqrt{1.301288503^3}}$$
$$= +0.913344637$$
$$3x = +0.818753074$$
$$x = +0.272917691$$
$$\sinh x = +0.276318334$$
$$z = m \sinh x$$
$$= 2\sqrt{\frac{\gamma}{3}} \sinh x$$
$$= 2\sqrt{\frac{1.301288503}{3}}(0.276318334)$$
$$= 0.363970234$$
$$= \tan 20^o$$

Check,

$$z^3 + 1.301288503z - 0.521846994 = 0$$
$$(0.363970234)^3 + 1.301288503(0.363970234) - 0.521846994 = 0$$
$$0.048216713 + 0.47363028 - 0.521846994 = 0$$
$$+0.521846994 - 0.521846994 = 0$$

13.4. Resolution via the Cubic Properties of an Ellipse.

In order to better demonstrate the *far-reaching implication* of CRT, independent of present day *Cubic Resolutions*, its overall philosophy is hereby further extended to a *new resolution* via *the Cubic Properties of an Ellipse*, complete with its own innate geometry, as follows:

Consider the ellipse shown in *Figure 12* with major and minor axes of lengths *a* and *b*, respectively; where point A is placed at the ellipse upper juncture with the y-axis, and to which right triangle ABC is adjoined with the following properties:

$\overline{AB} = a$ *(Hence point B represents a focus)*

$\overline{AC} = 1$

$\angle OAB = \theta$

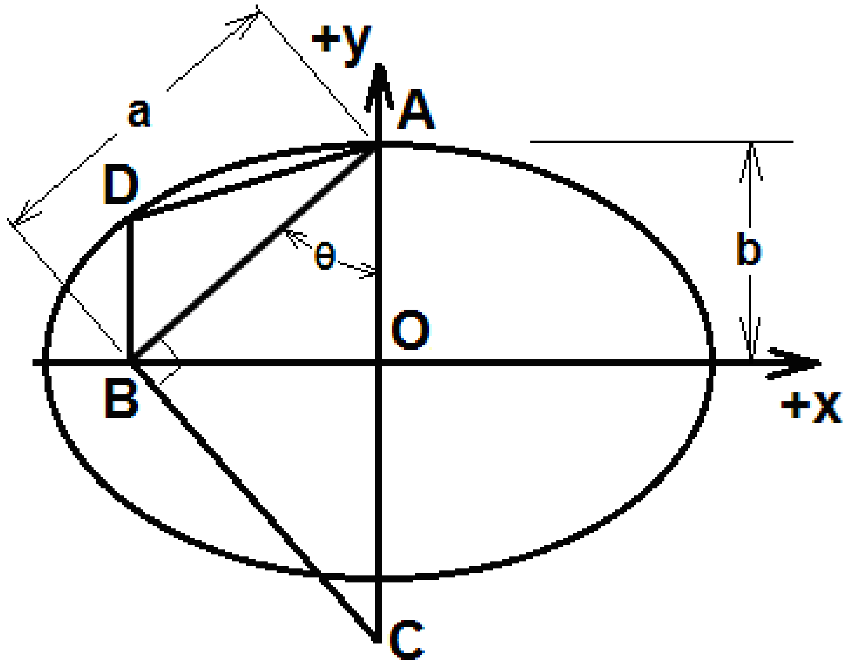
Then,

$$\cos\theta = \frac{\overline{AB}}{\overline{AC}} = \frac{a}{1} = \frac{\overline{OA}}{\overline{AB}} = \frac{b}{a}$$

Or,

$$b = a^2 = \overline{OA}$$

Figure 12. Cubic Properties of the Ellipse .



Now, the equation for an ellipse is given below:

$$1 = (\frac{x}{a})^2 + (\frac{y}{b})^2 \text{ Footnote 1}$$

$$= \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$= \frac{x^2}{b} + \frac{y^2}{b^2}$$

Multiplying each side of the equation by "b^2" gives:

$$b^2 = bx^2 + y^2$$

When,

$$x_B = \overline{OB}$$
$$= \overline{AB} \sin\theta$$
$$= a\sin\theta$$
$$= x_D$$

$$x_D{}^2 = a^2 \sin^2\theta$$
$$= b(1 - \cos^2\theta)$$
$$= b(1 - a^2)$$
$$= b(1 - b)$$
$$= b - b^2$$

1. CRC Standard Mathematical Tables 12th Printing; Charles D. Hodgman, M.S. - Editor in Chief; The Chemical Rubber Company, Cleveland Ohio; 1964; page 414.

TRUE SCANS

Then,

$$b^2 = bx_D{}^2 + y_D{}^2$$
$$= b(b - b^2) + y_D{}^2$$

Or,

$$0 = -b^3 + y_D{}^2$$
$$b^3 = y_D{}^2$$

Which equates to,

$$a^3 = y_D = \overline{BD}$$

Since alternate interior angles are equivalent:

$$\angle OAB = \angle ABD = \theta$$

Then, by the *Law of Cosines*:

$$\overline{AD}^2 = \overline{BD}^2 + \overline{AB}^2 - 2(\overline{BD})(\overline{AD})\cos(\angle ABD)$$
$$= y_D{}^2 + a^2 - 2(y_D)(a)\cos\theta$$
$$= (a^3)^2 + a^2 - 2(a^3)(a)(a)$$
$$= b^3 - 2ab^2 + b$$

Check via Pythagorean Theory:

$$\overline{AD}^2 = \overline{OB}^2 + (\overline{OA} - \overline{BD})^2$$
$$= x_D{}^2 + (b - y_D)^2$$
$$= (b - b^2) + (b - a^3)^2$$
$$= b - b^2 + b^2 - 2ba^3 + a^6$$
$$= b^3 - 2ab^2 + b$$

Letting $\overline{AD}$ be represented as "c", the above equation becomes:

$$c^2 = b^3 - 2ab^2 + b$$

Lastly, re-arranging above terms results in the following *Generalized Cubic Equation* [Ref. *Equation 32*] with α equal to unity and z represented by b:

Equation 44*. Cubic Elliptical Relationship.*

$$b^3 - 2ab^2 + b - c^2 = 0$$

In this case, *Equation 44* is of cubic form. However, when b becomes represented by a^2, the following sixth order equation results:

Equation 45*. Sixth Order Elliptical Relationship.*

$$a^6 - 2a^5 + a^2 - c^2 = 0$$

Hence the *Cubic Properties of an Ellipse (Ref. Figure 12)* are suitable for resolving 3rd and 6th order equations as follows:

Given the equation,

$$z^3 - 1.94874013z^2 + z - 0.048634409 = 0$$

Since the coefficient of the third term is equal to unity, in order to determine whether this given *Generalized Cubic Equation* (ref. *Equation 32*) qualifies as a *Cubic Elliptical Relationship*, determine the value of "a" by setting its β coefficient equal to -2a (ref. *Equation 44*) as follows:

$$\beta = -1.94874013 = -2a$$

$$\frac{1.94874013}{2} = a$$

$$0.974370064 = a = \cos 13^o$$

$$z = b = a^2 = 0.949397023$$

$$b^3 - 2ab^2 + b - c^2 = 0 \qquad \text{[ref. } \textit{Equation 44}\text{]}$$

$$(0.949397023)^3 - 2(0.974370064)(0.949397023)^2 + 0.949397023 - 0.048634409 = 0$$

$$0.855743476 - 1.756506088 + 0.949397023 - 0.048634409 = 0$$

$$1.805140499 - 1.805140499 = 0$$

$$0 = 0$$

Hence the *given equation* qualifies as a *Cubic Elliptical Relationship* with its first root being $z_1 = 0.949397023 = (\cos 13^o)^2$.

In order to verify that such given equation resides in a domain which may be resolved by the CRT:

Equation 42 is selected from the excerpt since it matches the above format,

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0$$

While considering u terms to be synonymous with z values, respective coefficients are compared to return the following results,

$$V = \frac{-1.94874013}{3} = -0.649580043$$

$$f = \pm\sqrt{V^2 - \frac{1}{3}} = \pm 0.297692625$$

$$\psi = \frac{V^3 - 3f^2V + 0.048634409}{2f^3} = \mp 0.999923457 = \cos(6\omega)$$

Secondly, resolve the unknown equation given above as follows:

$$6\omega = 179.2910867^o; 180.7089133^o; (179.2910867^o + 360^o)$$

Or,

$$6\omega = 0.708913326^o; 359.2910867^o; (0.708913326^o + 360^o)$$

$$2\omega = \frac{179.2910867^o}{3}; \frac{180.7089133^o}{3}; \frac{539.2910867^o}{3}$$
$$= 59.76369556^o; 60.23630444^o; 179.7636956^o$$

Or,

$$2\omega = \frac{0.708913326^o}{3}; \frac{359.2910867^o}{3}; \frac{360.708913326^o}{3}$$

$$= 0.236304442^o; 119.7636956^o; 120.236304442^o$$

$$\cos(2\omega) = +0.503567477; +0.496424017; -0.999991495$$

Or,

$$\cos(2\omega) = +0.999991495; -0.496424017; -0.503567477$$

$$2f\cos(2\omega) = 2(0.297692625)(+0.503567477; +0.496424017; -0.999991495)$$
$$= +0.299816648; +0.295563537; -0.595380186$$
$$= \ell$$

Or,

$$2f\cos(2\omega) = 2(-0.297692625)(0.999991495; -0.496424017; -0.503567477)$$
$$= -0.595380186; +0.295563537; +0.299816648$$
$$= \ell$$

Now since,

$$u = \ell - V$$
$$= (+0.299816648; +0.295563537; -0.595380186) - (-0.649580043)$$
$$= +0.949396691; +0.94514358; +0.054199857$$
$$= z_1; z_2; z_3$$

Or,

$$u = \ell - V$$
$$= (-0.595380186; +0.295563537; +0.299816648) - (-0.649580043)$$
$$= +0.054199857; +0.94514358; +0.949396691$$
$$= z_3; z_2; z_1$$

Check,

$$z_1^{\ 3} - 1.94874013 z_1^{\ 2} + z_1 - 0.048634409 = 0$$
$$b^3 - 1.94874013 b^2 + b - 0.048634409 = 0$$

$$(a^2)^3 - 1.94874013(a^2)^2 + (a^2) - 0.048634409 = 0$$
$$a^6 - 1.94874013a^5 + a^2 - 0.048634409 = 0$$
$$a^6 - 2(\cos 13^o)a^5 + a^2 - 0.048634409 = 0$$
$$(\cos 13^o)^6 - 2(\cos 13^o)(\cos 13^o)^4 + (\cos 13^o)^2 - 0.048634409 = 0$$
$$(\cos 13^o)^6 - 2(\cos 13^o)^5 + (\cos 13^o)^2 - 0.048634409 = 0$$
$$(0.974370064)^6 - 2(0.974370064)^5 + (0.974370064)^2 - 0.048634409 = 0$$
$$0.855743476 - 1.75650609 + 0.949397023 - 0.048634409 = 0$$
$$1.805140499 - 1.805140499 = 0$$
$$0 = 0$$

$$z_2{}^3 - 1.94874013z_2{}^2 + z_2 - 0.048634409 = 0$$
$$(0.94514358)^3 - 1.94874013(0.94514358)^2 + 0.94514358 - 0.048634409 = 0$$
$$0.844293345 - 1.740802517 + 0.054199857 - 0.048634409 = 0$$
$$1.789436925 - 1.789436926 = 0$$
$$0 = 0$$

$$z_3{}^3 - 1.94874013z_3{}^2 + z_3 - 0.048634409 = 0$$
$$(0.054199857)^3 - 1.94874013(0.054199857)^2 + 0.054199857 - 0.048634409 = 0$$
$$0.000159218 - 0.005724666 + 0.054199857 - 0.048634409 = 0$$
$$0.054359075 - 0.054359075 = 0$$
$$0 = 0$$

As a further check, z_2 and z_3 are verified as follows:

Where,

$$z^3 - 1.94874013z^2 + z - 0.048634409 = (z - z_1)(z - z_2)(z - z_3)$$
$$= z^3 - (z_1 + z_2 + z_3)z^2 + [z_1(z_2 + z_3) + z_2 z_3]z - z_1 z_2 z_3$$

Comparing like terms establishes the following two relationships:

$$-(z_1 + z_2 + z_3) = -1.94874013$$
$$z_1 + z_2 + z_3 = 1.94874013$$
$$z_2 + z_3 = 1.94874013 - z_1$$
$$= 0.999343106$$

$$-z_1 z_2 z_3 = -0.048634409$$
$$z_2 z_3 = \frac{0.048634409}{z_1}$$
$$= 0.051226628$$

Then,

$$z_3 = 0.999343106 - z_2$$

And,

$$z_2 z_3 = 0.051226628$$
$$z_2(0.999343106 - z_2) = 0.051226628$$
$$0.999343106 z_2 - z_2^{\ 2} = 0.051226628$$

Completing the Square renders:

$$z_2^{\ 2} - 0.999343106 z_2 + (\frac{0.999343106}{2})^2 = -0.051226628 + (\frac{0.999343106}{2})^2$$
$$(z_2 - \frac{0.999343106}{2})^2 = 0.198445032$$
$$z_2 - \frac{0.999343106}{2} = \pm\sqrt{0.198445032}$$
$$z_2 - 0.499671553 = \pm 0.445471697$$
$$z_2 = 0.499671553 \pm 0.445471697$$
$$= 0.94514325; 0.054199856$$

$$z_3 = 0.999343106 - z_2$$
$$= 0.054199856; 0.94514325$$

The above given equation also portrays the following sixth order equation:

$$a^6 - 2a^5 + a^2 - c^2 = 0 \qquad \textit{[Ref. Equation 45]}$$

$$a^6 - 1.94874013a^4 + z - 0.048634409 = 0$$

Such postulated equation is resolved in the same manner by first verifying that is qualifies as a bonafide *Sixth Order Elliptical Relationship* as follows:

$$\beta = -1.94874013 = -2a$$
$$\frac{1.94874013}{2} = a$$
$$0.974370064 = a = \cos 13^o$$

$$a^6 - 2a^5 + a^2 - c^2 = 0$$
$$(0.974370064)^6 - 2(0.974370064)(0.974370064)^4 + (0.974370064)^2 - 0.048634409 = 0$$
$$(\cos 13^o)^6 - 2(\cos 13^o)(\cos 13^o)^4 + (\cos 13^o)^2 - 0.048634409 = 0$$
$$(\cos 13^o)^6 - 2(\cos 13^o)^5 + (\cos 13^o)^2 - 0.048634409 = 0$$
$$(0.974370064)^6 - 2(0.974370064)^5 + (0.974370064)^2 - 0.048634409 = 0$$
$$0.855743476 - 1.756506088 + 0.949397023 - 0.048634409 = 0$$
$$1.805140499 - 1.805140499 = 0$$
$$0 = 0$$

TRUE SCANS

Note that the above analysis was conducted without having to impose the *present day measure* of first having to *transform* the given *Cubic Equation* into another devoid of a second order term.

Hence the *given equation* qualifies as a *Sixth Order Elliptical Relationship* with its first root being $z_1 = 0.974370064 = \cos 13^o$.

The six roots of the *given equation* are obtained by employing roots determined above, as follows:

Where,

$$b_1; b_2; b_3 = +0.949396691; +0.94514358; +0.054199857$$

Since,

$$a = \pm\sqrt{b}$$

$$a_{11} = +\sqrt{b_1} = +\sqrt{0.949396691} = +0.974369894$$

$$a_{12} = -\sqrt{b_1} = -\sqrt{0.949396691} = -0.974369894$$

$$a_{21} = +\sqrt{b_2} = +\sqrt{0.94514358} = +0.972184951$$

$$a_{22} = -\sqrt{b_2} = -\sqrt{0.94514358} = -0.972184951$$

$$a_{31} = +\sqrt{b_3} = +\sqrt{0.054199857} = +0.232808627$$

$$a_{32} = -\sqrt{b_3} = -\sqrt{0.054199857} = -0.232808627$$

TRUE SCANS

13.5. Resolution of *Generalized Cubic Equations* of the *Form* $z^3+\delta=0$.

A *Generalized Cubic Equation (GCE)* whose β and γ terms are equal to *zero* assumes the following *form*:

$$\alpha z_R{}^3 + \beta z_R{}^2 + \gamma z_R + \delta = 0 \quad \text{[Ref. Equation 32]}$$

$$1z_R{}^3 + 0z_R{}^2 + 0z_R + \delta = 0$$

$$z_R{}^3 + \delta = 0$$

Where,

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \quad \text{[Ref. Equation 36]}$$

$$= \frac{\delta - 0}{1 - 0}$$

$$= \delta$$

Then,

$$z_R{}^3 + \zeta = 0$$

$$z_R{}^3 = -\zeta = (R\tan\theta)^3$$

$$z_R = \sqrt[3]{-\zeta} = R\tan\theta$$

Equation 46. *'R' Determination for a GCE Devoid of its 2nd & 3rd Terms.*

$$z_R = \sqrt[3]{-\zeta} = R\tan\theta$$

An example of how *Equation 46* is applied is afforded below for the particular condition when:

$$\xi = \tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^o$$

$$\theta = 18.43494882^o$$

$$\tan\theta = 1/3$$

Where,

$$z_R = \sqrt[3]{-\zeta} \quad \text{[Ref. Equation 46]}$$

$$= \sqrt[3]{-\frac{13}{9}(\frac{3}{3})}$$

$$= -\frac{1}{3}\sqrt[3]{39}$$

$$= -\frac{1}{3}(3.391211443) = R\tan\theta = \frac{R}{3}$$

$$z_R = -1.130403814$$

$$3.391211443 = R$$

The *equation form* $z_R{}^3 + \zeta = 0$ also applies to *GCE's* when $\gamma = -\beta z_R$

TRUE SCANS

SECTION 14. CURVE MAPPING.

Curve Mapping characterizes *relative movement* through *alterations* evidenced within ***coefficient structures***, also termed the ***fabric of mathematical functions***, which serve to independently specify a *stationary curve shape* which exhibits a *singular* location in space.

Different perspectives of such *unique curve* in space are gained, or realized by applying a *relativistic approach* which permits a *mobile origin* to roam, or move about to other *pre-selected points* upon an orthogonal *grid pattern*.

Curve Mapping mathematically operates upon just one particular *coefficient structure*, or ***equation format*** at a time. As indicated above, this consists of a suite, or *multiplicity of individual mathematical functions* that all describe the *same, exact curve shape*. Their *fabric* changes depending upon which particular viewpoint of the unique curve shape is perceived at any given time. Hence, varying perspectives return different perceptions of the same singular entity! The *mobile origin* concept allows this to occur, thereby *influencing* the particular constitution of the *fabric*, or *coefficient structure*, depending upon relative location.

Each ***equation format*** consists of a *set, or family* of *coefficient permutations* comprised of intrinsic ***RST terminology***. Hence, a gateway for ***Equation Sub-element*** *categorizations* becomes realized!

The following *Parabolic* and *Generalized Cubic Function coefficient structures* are to be examined for reasons purported in *Section 1*.

$\mathrm{ax}^2 + \mathrm{bx} + \mathrm{c} = \mathrm{y}$ (*Ref. Section 14.1*)

$\alpha z^3 + \beta z^2 + \gamma z + \delta = y$ (*Ref. Section 14.2*)

Below it is explained just how such overall *fabric*, or *coefficient structure houses* intrinsic *root* and *mobile origin* ***location*** information, which may be deciphered in terms of *algorithmic interpretations* posed below.

The *Curve Mapping* process is presented with hopes, or aspirations of availing a deeper understanding into ***Equation Sub-element*** categorizations.

TRUE SCANS

14.1. Parabolic Curve Mapping.

Figure 13 depicts a family *of Parabolic Curves*, all of which are *superimposed* directly upon one another and feature *identical curve shape*.

Such *multiplicity of associated curves*:

- Satisfy all constraints imposed upon them by the *General Parabolic Function*:

  $$ax^2 + bx + c = y$$

- Subscribe to the *singular curve shape*:

  $$ax^2 = y$$

14.1.1. Singularity Proof.

Figure 13 introduces *three points* as follows:

- Point *O* represents a ***primary origin*** that is located at the *low point* of the following *given Function*:

  $$ax_O^2 = y_O$$

  This origin resides at $x_O = y_O = 0$ such that,

  $$a(0)^2 = 0$$

  The subscripts x_O and y_O shown above denote respective horizontal and vertical displacements between point *O*, the primary origin, and various other locations upon this curve

- Point *A*, positioned directly above, or below, Point *O* an arbitrary distance y_M, locates a ***relative*** *secondary* origin from which the *unique Parabolic Curve set* described above may be viewed. Since there exists *only* a *vertical* displacement between relative origins:

  $$x_O = x_A$$

  $y_O = y_A - y_M$ *Given that y_M is a negative quantity*

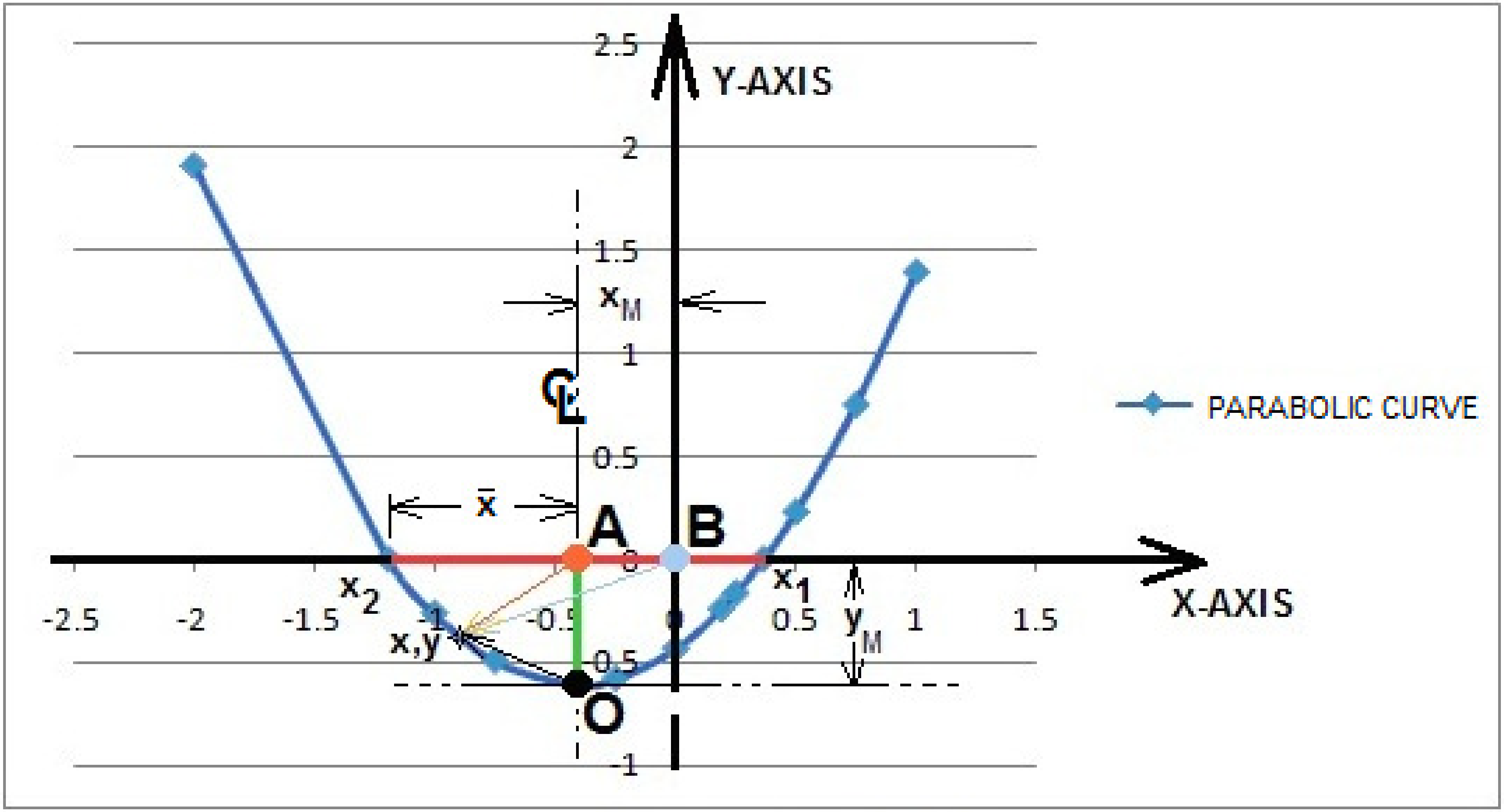
  Then, from above:

  $$ax_O^2 = y_O$$

  $$ax_A^2 = y_A - y_M$$

TRUE SCANS

Figure 13. Quintessential Relationship for Parabolic Curve Mapping.



- Point *B*, positioned directly to the right, or left, of Point *A* an arbitrary offset of x_M, locates a *relative tertiary* origin from which this *singular Parabolic Curve* also may be viewed. From *Figure 13* it is evidenced that $x_B = x_A + x_M$, given that x_M also is displayed as a negative quantity such that:

$$x_A = x_B - x_M$$

$$x_A^2 = x_B^2 - 2x_M x_B + x_M^2$$

$$ax_A^2 + y_M = ax_B^2 - 2ax_M x_B + a x_M^2 + y_M$$

From above, since:

$$ax_A^2 = y_A - y_M$$

It follows that,

$$ax_A^2 + y_M = y_A$$

$$= y_B$$

Equating the above two results renders,

$$ax_B^2 - (2ax_M)x_B + (a x_M^2 + y_M) = y_B$$

For any assigned values of *a*, x_M and y_M, respectively, let:

$$b = -2ax_M$$

$$c = ax_M^2 + y_M$$

Then,

$$ax_B^2 + bx_B + c = y_B$$

Respective x_M and y_M values for this resulting *Parabolic Function*, can be determined from any *given coefficients* as follows:

$$x_M = -\frac{b}{2a}$$

$$y_M = c - ax_M^2$$

$$= c - a\left(-\frac{b}{2a}\right)^2$$

$$= c - \frac{b^2}{4a}$$

TRUE SCANS

Of note, the ***curve*** for this given *Parabolic Function (Ref. Figure 13)*, along with *two others* which are associated with the *transforms* derived above, coincide; i.e., all three *collapse*, or become *superimposed* upon one another. Such *three functions* are re-listed as follows:

$$ax_O^2 = y_O$$

$$ax_A^2 + y_M = y_A$$

$$ax_B^2 + bx_B + c = y_B$$

The very fact that *all three functions* represent the same *exact, singular curve* is demonstrated below:

- As y_A is set equal to zero:

$$ax_A^2 + y_M = y_A$$

$$ax_A^2 + y_M = 0$$

$$ax_A^2 = -y_M$$

$$x_A = \pm\sqrt{-\frac{y_M}{a}}$$ *[Ref. points x_1 and x_2 in Figure 13]*

- As y_A is set equal to zero:

$$y_O = y_A - y_M$$

$$= 0 - y_M$$

Or,

$$y_O = -y_M$$

Lastly, via substitution:

$$ax_O^2 = y_O$$

$$ax_O^2 = -y_M$$

$$x_O = \pm\sqrt{-\frac{y_M}{a}}$$

$$= x_A$$

This is to be expected since $x_O = x_A$ holds for all points upon the *given Parabolic Function*.

In other words, when y_A equals zero, it corresponds to a relative location on the *given Parabolic Curve* where $y_O = -y_M$. For ***this elevation only***, which happens to coincide with the x-axis shown in *Figure* 13, the two x_A values much coincide with respective x_O values (since $x_O = x_A$). Hence two *identical points* are mapped out.

TRUE SCANS

Moreover, the *step function* $y_A = y_O + y_M$ signifies that for any value of y_O, there exists a respective value of y_A that always is equal to such y_A value plus y_M.

Since x_A shares the same value as x_O anywhere along the *given Parabolic Curve*, it becomes obvious that the first two functions (of the three designated above) map out ***identical curves*** because for any selected location x_O and y_O on the *given Parabolic Curve*, another location is mapped out such that $x_A=x_O$ and $y_A = y_O + y_M$, thereby designating *an exact curve fit*.

This demonstrates that all respective *intermediate points* on the two curves also match up, once compensating for the relative displacement of origins.

For example, location $x_O = y_O = 0$ resides a distance y_M below the *relative x-axis* depicted in *Figure 13*.

Then,

$$y_A = y_O + y_M$$
$$= 0 + y_M$$
$$= y_M$$

Since this vertical distance, calculated above, is measured off from the *second origin* (which resides at point *A*), its corresponding location, as designated below, superimposes directly upon the given location of $x_O = y_O = 0$ as determined from the initial origin which resides at point O:

$$x_A = 0$$
$$y_A = y_M$$

Moreover, considered to be an *algebraic variable*, the term y_M can assume an *infinite* number of values.

- Its associated *compensating* term y_A is determined via the equation $y_O + y_M - y_A = 0$ (see above). y_A is described as *compensating* because it adjusts for values of y_O which satisfy $a{x_O}^2 = y_O$.
- Its other associated *compensating* term *c* is determined via the equation below. The term *c* is described as *compensating* because it adjusts for values of $b^2/(4a)$ which satisfy the function $ax^2 + bx + c = y$

$$c = y_M + \frac{b^2}{4a}$$

Likewise, the *algebraic variable c* also can assume an *unlimited* number of values. Its associated, *compensating* term *b* is determined via $y_M - c + \frac{b^2}{4a} = 0$ (see above) for each and every *a* and *c* set of values that are proposed.

Based upon the above analysis, attempting to pictorially depict the three identically shaped *Parabolic Functions* described above with respect to a *common origin* would merely superimpose them on top of one another.

Again, this is validated *mathematically* by transforming the function whose origin resides at *Point B*, cited below, to one whose origin resides at *point A*, and lastly to one whose origin resides at *Point O* as follows; thereby showing all three functions to be absolutely identical representations of one another:

$$ax_B{}^2 + bx_B + c = y_B$$

Where

$x_B = x_A + x_M$ *(Ref. Figure 13)*

$y_B = y_A$ *(Ref. Figure 13)*

Via substitution,

$$a(x_A + x_M)^2 + b(x_A + x_M) + c = y_A$$

$$a(x_A)^2 + (2ax_M)x_A + a(x_M)^2 + bx_A + bx_M + c = y_A$$

Such that,

$$b = -2ax_M$$

$$c = y_M + \frac{b^2}{4a}$$

$$a(x_A)^2 + (2ax_M)x_A + a(x_M)^2 - (2ax_M)x_A - (2ax_M)x_M + y_M + \frac{b^2}{4a} = y_A$$

$$a(x_A)^2 + a(x_M)^2 - 2a(x_M)^2 + y_M + \frac{4a^2}{4a}(x_M)^2 = y_A$$

$$a(x_A)^2 + a(x_M)^2 - 2a(x_M)^2 + y_M + a(x_M)^2 = y_A$$

$$a(x_A)^2 + y_M = y_A \quad \textit{Q.E.D.}$$

Lastly, where

$x_A = x_O$ *(Ref. Figure 13)*

$y_A = y_O + y_M$ *(Ref. Figure 13)*

$$a(x_O)^2 + y_M = y_O + y_M$$

$$a(x_O)^2 = y_O \quad \textit{Q.E.D}$$

TRUE SCANS

Alternatively, the three functions may be simultaneously mapped so that they instead become illustrated besides one another, simply by overlooking their respective *origin nomenclatures* and thereby plotting them with respect to a *common origin*, say the one residing at *point A*. Accordingly, such three *Parabolic Curves*, devoid of *origin nomenclature* are portrayed in *Figure 14* and accorded the following designations:

- *A-curve* -- The *curve* whose *low point* resides at *point A*, and is of the form $ax^2 = y$
- *O-curve* -- The curve whose low point resides directly below that of the *A-curve*, and is of the form $ax^2 + y_M = y$
- *B-curve* -- The curve also has its low point coinciding with the x-axis, but is located at a distance of x_M away from point *A*, and is of the form $ax^2 + bx + c = y$

The primary objective of *Figure 14* is to demonstrate that all three curves are *identical* by virtue of the fact that:

- Vertical steps of $\Delta = y_M$ occur to the *A* curve for respective incremental changes applied to x_O. Then,

  $$y_A \text{ (at } x_A = x_O) = y_O + y_M = y_O + \Delta$$

- Horizontal steps of $\delta = x_M$ occur to the *B* curve for respective incremental changes applied to y_A. Then,

  $$x_B \text{ (at } y_B = y_A) = x_A + x_M = x_A + \delta$$

Such assignments signify that the two upper curves depicted in *Figure 14* may be displaced until they exactly coincide with the lower *given Parabolic Curve*. Naturally the amount of their relocation is exactly the distance between respective origins.

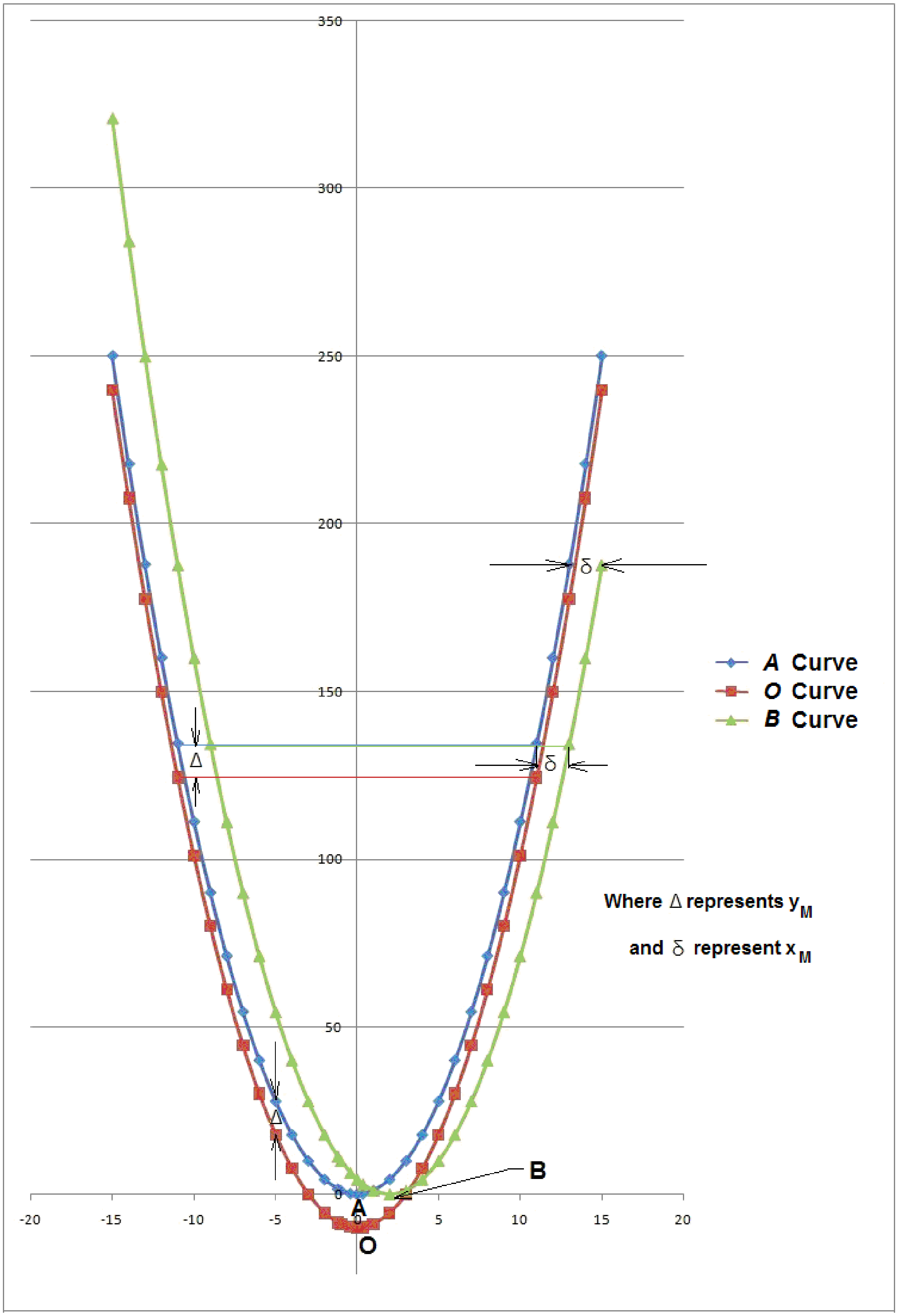
So, once this compensation for origin difference becomes accounted for, the three curves become one, or coincide.

In conclusion, a *given Parabolic Function* along with *two transforms* derived from it produce a total of three *identically shaped Parabolic Curves* ***all*** of which *occupy the same exact coordinates*, where each merits its *own* independent *perspective*, or *point of origin*.

Accordingly, it now becomes possible to *associate* given *Parabolic Curves* with a plethora of other *Parabolic Functions* which exhibit their same exact shape.

TRUE SCANS

Figure 14. Identical Curves Intended to be Superimposed upon Each Other.



14.1.2. The Algorithm.

Figure 13 now is to be viewed as a grid of *potential* ***origin relocations*** away from the *low point* of any given *Parabolic Curve*. Such *stationary curve* then can be perceived from *Point B,* a *relocated origin*, with the y-axis crossing it.

Consider further that such *arbitrarily selected origin* upon this *grid* harbors a unique *Parabolic Function* capable of characterizing the *given Parabolic Curve on its own*.

Such conjecture promotes a possibility to afford *unique solutions* based upon any vantage point, or *perspective* that is desired. Conversely, the entire *grid* then would prefer an infinite *number of solutions* to characterize such *given Parabolic Curve*, all of which belong to a *family* that comprises the function $ax^2 = y$.

One might ask why such *pretension* is necessary in the first place, considering that a *Parabolic Curve* simply may be resolved in terms of the variable *x*, after *characterizing it at its low point* by determining the unknown quantity *a* through any of the curve's plotted points as follows:

$$a = y/x^2$$

The answer is that only *one set of solutions* is afforded. However, such set is limited because there is no accounting of its association with all of the other solutions which exist. And it may very well be another of these solutions which really answers what the problem solver is searching for – a solution rendered from another perspective, or completely independent *origin system*!

Moreover, such pretext conveys insight into a ***Equation Sub-element*** association that connects families of *Parabolic Function* coefficient relationships together. More specifically, it relates any given *Parabolic Curve*, not only to any *arbitrarily selected location* on a *grid*, but to a distinct characterization of the tan θ.

The very essence of this *algorithm*, therefore, lies in the fact that it *perceives* the *given Parabolic Curve* from *different elevations* or perspectives that intersect it at respective *pairs of locations*, termed root sets, each of which is described by a unique suite of *trigonometric solutions* which all, nevertheless, assume the form of the *General Parabolic Function*:

$$ax^2 + bx + c = y$$

TRUE SCANS

The algorithm for establishing ***Parabolic Curve Mapping*** relies upon certain *quintessential elements*, or properties of *Parabolic Functions* as denoted below *(Ref. Figure 13)*:

- $\boldsymbol{y_M}$ depicts a *vertical* offset from the low point (or high point) of the *given Parabolic Curve* to any other arbitrarily selected height elevation, or latitude. *Most importantly*, it represents a *curve characteristic* that can be easily identified and measured, and therefore qualifies as a *parabolic curve* *property*
- $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ represent *distinct roots* for the *Parabolic Curve* relative to any arbitrarily selected value of y_M. They occur at *parabolic curve* intersection points with a horizontal line which is drawn through the end of the y_M offset which is not connected to the *parabolic curve's* low point
- The ***origin*** is a point selected upon this horizontal projection which connects x_1 to x_2. *Most importantly*, for the particular origin specified, x_1 and x_2 represent respective distances away from the origin.
- $\boldsymbol{x_1 - x_2}$ identifies the distance between roots x_1 and x_2. It also may be viewed as the horizontal length between *Parabolic Curve* intersection points at any arbitrarily selected height y_M above the curve's low point (or below it high point). In this case, $+x_1$ signifies a root which is located to the right of the origin and $-x_2$ signifies a root located to the left of the y-axis. As a measurable feature, $x_1 - x_2$ also must be listed as a *Parabolic Curve* *property*
- $\bar{x}$, as specified below reflects a *symmetrical aspect* of the *given Parabolic Curve*. This is easily verified by its defining function $ax^2 = y$ such that:

  $$\bar{x} = \pm\sqrt{y/a}$$

  This above equation constitutes dual quantities of x for every value of y postulated, located identical distances to the left and to the right from the centerline of the curve, respectively – which designates symmetry.

  Above, the term $\bar{x}$ is used to denote a horizontal distance which runs from *any* point on the *parabolic curve* to the centerline of that curve, as represented by an imaginary line which runs vertically through point *O*.

Accordingly,

$$y = a(\bar{x})^2 = -y_M$$

Now, going back to *Figure 13*, a typical *Parabolic Curve* is represented whose *low point* is located at a distance, or coordinate x_M and y_M, respectively from the origin. In this case, both x_M and y_M are to be considered as *negative quantities* since they are measured to the *low point* of the *Parabolic Curve*, which lies both to the left and below the specified origin.

Because of this symmetry, $\bar{x}$ must be equal to one-half of the *property* $(x_1 - x_2)$. However, from *Figure 13*, it is shown that such length also is equal to $x_1 - x_M$ (with x_M designated as a negative quantity). Accordingly,

$$x_1 - x_M = ½(x_1 - x_2)$$
$$2x_1 - 2x_M = x_1 - x_2$$
$$-2x_M = -(x_1 + x_2)$$

Or,

$$x_M = ½(x_1 + x_2)$$

Where it was determined that:

$$x_M = -\frac{b}{2a} \qquad \textit{(Ref. Section 14.1.1)}$$

So,

$$½(x_1 + x_2) = -\frac{b}{2a}$$
$$a(x_1 + x_2) = -b$$

Or,

$$b = -a(x_1 + x_2)$$

Furthermore, from:

$$y = -y_M = a(\bar{x})^2$$
$$= a(\frac{x_1 - x_2}{2})^2$$

But, earlier it was determined that:

$$y_M = c - b^2/4a\,)$$
$$y_M + b^2/4a = c$$
$$-a(\frac{x_1 - x_2}{2})^2 + b^2/4a = c$$

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As such,

$$
\begin{aligned}
c &= -a(\frac{x_1 - x_2}{2})^2 + \frac{b^2}{4a} \\
&= -a(\frac{x_1 - x_2}{2})^2 + \frac{[-a(x_1 + x_2)]^2}{4a} \\
&= \frac{a}{4}(-x_1{}^2 + 2x_1x_2 - x_2{}^2 + x_1{}^2 + 2x_1x_2 + x_2{}^2) \\
&= ax_1x_2
\end{aligned}
$$

To summarize, the coefficients *b* and *c* now both have become postulated in terms of x_1 and x_2 as follows:

- $b = -a(x_1 + x_2)$
- $c = ax_1x_2$

Check:

$$
\begin{aligned}
ax_1{}^2 + bx_1 + c &= 0 \\
ax_1{}^2 - a(x_1 + x_2)x_1 + ax_1x_2 &= 0 \\
x_1{}^2 - x_1{}^2 - x_1x_2 + x_1x_2 &= 0 \\
0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
ax_2{}^2 + bx_2 + c &= 0 \\
ax_2{}^2 - a(x_1 + x_2)x_2 + ax_1x_2 &= 0 \\
x_2{}^2 - x_1x_2 - x_2{}^2 + x_1x_2 &= 0 \\
0 &= 0
\end{aligned}
$$

Q.E.D.

Now further suppose that x_M and y_M are selected in such as manner that the *origin* is placed at a position, relative to the low point on the *Parabolic Curve* where *distances* measured horizontally from this origin to intersection points on the *curve* are as follows:

- $x_1 = \tan\theta$
- $x_2 = -1/\tan(2\theta)$

Such selection is merely arbitrary. That is, it meets the needs of this analysis by allowing efficient computation, but in no way represents the only selection that could be adopted.

Then at,

$$
\begin{aligned}
b &= -a(x_1 + x_2) \\
&= -a[\tan\theta - \frac{1}{\tan(2\theta)}] \\
&= -a(\tan\theta - \frac{1-\tan^2\theta}{2\tan\theta}) \\
&= a(\frac{1-3\tan^2\theta}{2\tan\theta})
\end{aligned}
$$

$$
\begin{aligned}
c &= ax_1x_2 \\
&= a\tan\theta[\frac{-1}{\tan(2\theta)}] \\
&= -a\tan\theta(\frac{1-\tan^2\theta}{2\tan\theta}) \\
&= \frac{a}{2}(\tan^2\theta - 1)
\end{aligned}
$$

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Check:

$$ax_1^{\,2} + bx_1 + c = 0$$

$$a\tan^2\theta + a\left(\frac{1-3\tan^2\theta}{2\tan\theta}\right)\tan\theta + \frac{a}{2}(\tan^2\theta - 1) = 0$$

$$\tan^2\theta + \frac{1}{2} - \frac{3}{2}\tan^2\theta + \frac{\tan^2\theta}{2} - \frac{1}{2} = 0$$

$$\frac{3}{2}\tan^2\theta + \frac{1}{2} - \frac{3}{2}\tan^2\theta - \frac{1}{2} = 0$$

$$0 = 0$$

$$ax_2^{\,2} + bx_2 + c = 0$$

$$a\left[-\frac{1}{\tan(2\theta)}\right]^2 + a\left(\frac{1-3\tan^2\theta}{2\tan\theta}\right)\left[-\frac{1}{\tan(2\theta)}\right] + \frac{a}{2}(\tan^2\theta - 1) = 0$$

$$a\left(\frac{\tan^2\theta - 1}{2\tan\theta}\right)^2 + a\left(\frac{1-3\tan^2\theta}{2\tan\theta}\right)\left(\frac{\tan^2\theta - 1}{2\tan\theta}\right) + \frac{a}{2}(\tan^2\theta - 1) = 0$$

$$\left(\frac{\tan^4\theta - 2\tan^2\theta + 1}{4\tan^2\theta}\right) + \left(\frac{\tan^2\theta - 1 - 3\tan^4\theta + 3\tan^2\theta}{4\tan^2\theta}\right) + \frac{2\tan^4\theta - 2\tan^2\theta}{4\tan^2\theta} = 0$$

$$\tan^4\theta - 2\tan^2\theta + 1 + \tan^2\theta - 1 - 3\tan^4\theta + 3\tan^2\theta + 2\tan^4\theta - 2\tan^2\theta = 0$$

$$3\tan^4\theta - 4\tan^2\theta + 1 + 4\tan^2\theta - 1 - 3\tan^4\theta = 0$$

$$0 = 0$$

In conclusion the coefficients for the equation $\mathrm{ax}^2 + \mathrm{bx} + \mathrm{c} = \mathrm{y}$ are determined to be as follows:

- $b = a\left(\frac{1-3\tan^2\theta}{2\tan\theta}\right)$
- $c = \frac{a}{2}(\tan^2\theta - 1)$

By setting x_1 and x_2 as follows, a *distinct point B* becomes located *(Ref. Figure 13)* which resides somewhere on the horizontal line which connects these above stated roots:

- $x_1 = \tan\theta$
- $x_2 = -1/\tan(2\theta)$

Now if it is desired to perform an analysis with regard to another location on this horizontal line besides point B, then the *end conditions* specified above simply do not apply. This issue is handled as follows:

Where the *two roots* to a *Parabolic Equation* are represented as,

$$(x - x_1)(x - x_2) = 0$$

$$x^2 - (x_1 + x_2)x + x_1 x_2 = 0$$

But,

$$ax^2+bx+c=0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0\text{, which leads to } \frac{b}{a}=-(x_1+x_2)$$

Then, for:

- $x_1=\tan\theta$
- $x_2=-1/\tan(2\theta)$

$$\frac{b}{a}=-(x_1+x_2)$$
$$=-[\tan\theta-\frac{1}{\tan(2\theta)}]$$
$$=-(\tan\theta-\frac{1-\tan^2\theta}{2\tan\theta})$$
$$=-(\frac{3\tan^2\theta-1}{2\tan\theta})$$
$$=\frac{1-3\tan^2\theta}{2\tan\theta}$$
$$=\frac{3-\tan^2\theta}{2\zeta}$$

- $x_1=\tan\theta$
- $x_2=-1/\tan\theta$

$$\frac{b}{a}=-(x_1+x_2)$$
$$=-(\tan\theta-1/\tan\theta)$$
$$=-\tan\theta+\frac{1}{\tan\theta}$$
$$=\frac{1-\tan^2\theta}{\tan\theta}$$

- $x_1=\tan\theta$
- $x_2=-\tan(2\theta)$

$$\frac{b}{a}=-(x_1+x_2)$$
$$=-[\tan\theta-\tan(2\theta)]$$
$$=-\tan\theta+\frac{2\tan\theta}{1-\tan^2\theta}$$
$$=\frac{\tan\theta+\tan^3\theta}{1-\tan^2\theta}$$
$$=\frac{\tan\theta+3\tan\theta-\zeta(1-3\tan}{1-\tan^2\theta}$$
$$=\frac{4\tan\theta-\zeta(1-3\tan^2\theta)}{1-\tan^2\theta}$$

Moreover,

$$\frac{b}{a}=-(x_1+x_2):$$

$$x_2=-(x_1+\frac{b}{a})$$

Or,

$$-x_2=x_1+\frac{b}{a}$$

Using this formula, $-x_2$ then may be calculated using the coefficients afforded in any given *Parabolic Equation*. This will enable resolution for any intermediate point located on the horizontal line which joins root x_1 to x_2.

Hence, determination of the proper term which should be applied to the x_2 root of any given *Parabolic Equation* is based upon the factor *b/a*, hereinafter to be identified as the *Selectivity Coefficient Ratio*.

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14.1.3. Application.

Table 18 lists ***Essential Algorithmic Relationships*** x_M and y_M for the *specific condition* when $x_1 = \tan\theta$ and $x_2 = -1/\tan(2\theta)$ when *eighteen values* of θ ranging from 5° to 85° in 5 degree increments and a value for θ of 83.0175° become applied.

Notice that *all rows* of its *third column* display the same arbitrarily assigned $a = 3$ *coefficient value*. From this, values of coefficients *b* and *c* are easily calculated as indicated in the two following sample cases:

1) For $\tan 5^{\circ} = 0.087488663$,

$$b = a(\frac{1-3\tan^2\theta}{2\tan\theta}) = 3[\frac{1-3(0.08748866)^2}{2(0.08748866)}] = 16.75137947$$ *(Ref. above table)*

$$c = \frac{a}{2}(\tan^2\theta - 1) = \frac{3}{2}[(0.087488663)^2 - 1] = -1.488518601$$

2) For $\tan 65^{\circ} = 2.144506921$:

$$b = a(\frac{1-3\tan^2\theta}{2\tan\theta}) = 3[\frac{1-3(2.144506921)^2}{2(2.144506921)}] = -8.950819655$$

$$c = \frac{a}{2}(\tan^2\theta - 1) = \frac{3}{2}(2.144506921)^2 - 1] = 5.398364898$$

The *roots* x_1 *and* x_2, along with values of x_M and y_M are established by applying respective equations listed at the top of *Table* 18.

For x = -12 and $\tan 65^{\circ} = 2.144506921$:

$$\begin{aligned} ax^2 + bx + c &= 3(-12)^2 + (-8.950819655)(-12) + 5.398364898 \\ &= 544.8082008 \\ &= y \end{aligned}$$

Figure 15*Figure 15* illustrates a family of eighteen resultant curves that each are associated with the eighteen respective rows of θ and tan θ values appearing in Table 18 such that:

- Its abscissa, or x-axis, displays x values where the subject curves are continuously charted between $-15 \le x \le +15$
- Its ordinate, or y-axis, displays values which characterize the sum of left-hand terms for the *Parabolic Function* $ax^2 + bx + c = y$

Table 19 renders a tabulation of its *plot points* where two specific calculations are afforded below for respective *b and c coefficient values* which were determined above:

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1) For x = 15 and $\tan 5^{o} = 0.087488663$,

$$\begin{aligned} ax^2 + bx + c &= 3(15)^2 + (16.75137947)(15) - 1.4885186 \\ &= 675 + 251.2706921 - 1.4885186 \\ &= 924.782173 \\ &= y \end{aligned}$$

TRUE SCANS

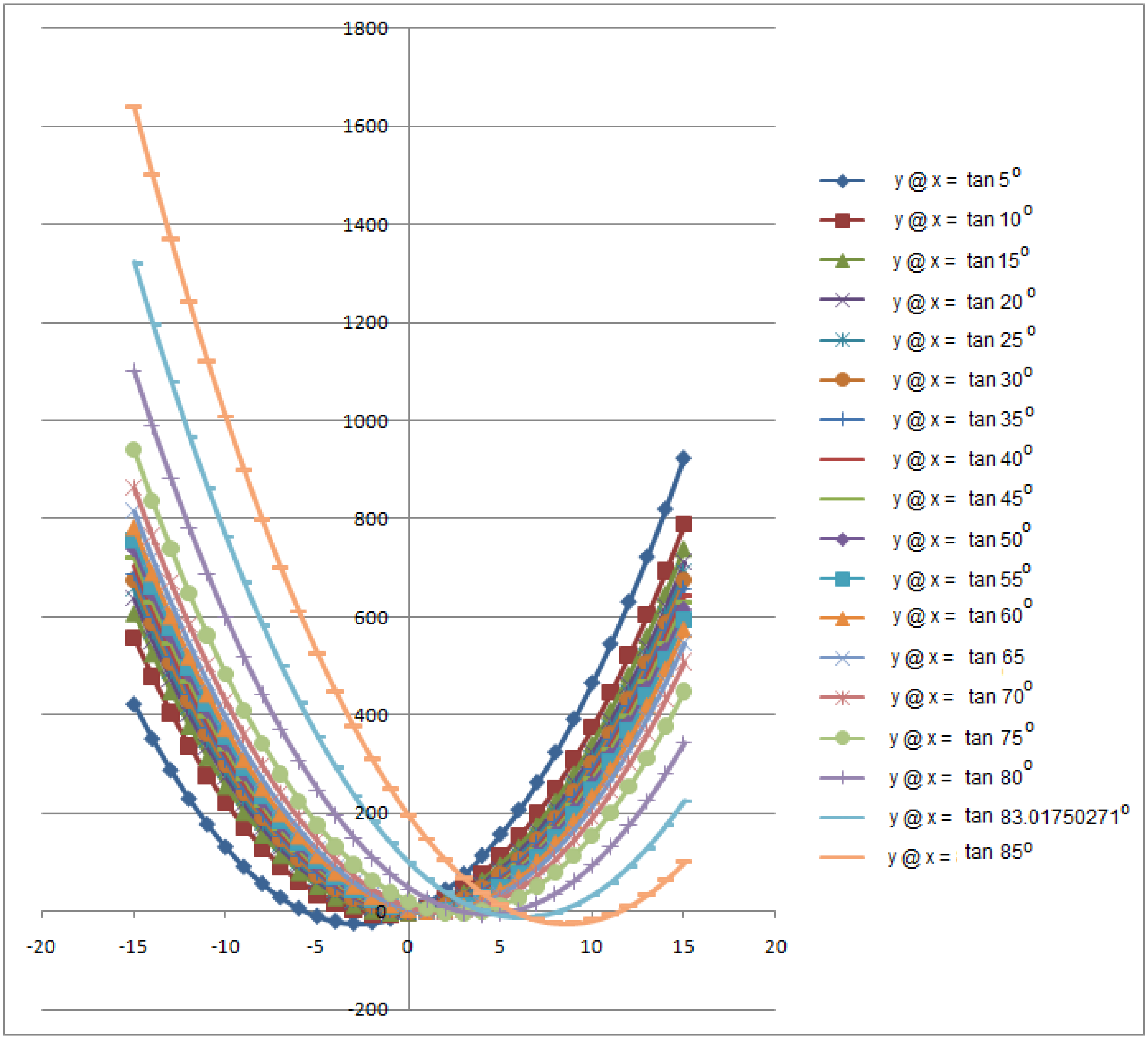
Table 18. Essential Algorithmic Relationships.

ANGLES	TANGENTS	*a*, *b*, and *c* COEFFICIENTS			QUADRATIC ROOTS		$x_M = -\frac{b}{2a}$	$y_M = ax_M^2 + bx_M + c$
		ARBITRARY COEFFICIENT	$b = a(\frac{1-3\tan^2\theta}{2\tan\theta})$	$c = \frac{a}{2}(\tan^2\theta - 1)$	$x_1, x_2 = [-b \pm \sqrt{b^2-4ac}]/2a$			
Θ (Deg)	tan θ	a	b	c	x_1	x_2	x_M	y_M
5	0.08748866	3	16.75137947	-1.488518601	0.087488663	-5.67128182	-2.791896578	-24.87257811
10	0.17632698	3	7.713451316	-1.453363194	0.176326981	-2.74747742	-1.285575219	-6.411474128
15	0.26794919	3	4.392304845	-1.392304845	0.267949192	-1.73205081	-0.732050808	-3
20	0.36397023	3	2.483350075	-1.301288503	0.363970234	-1.19175359	-0.413891679	-1.815207469
25	0.46630766	3	1.118375919	-1.173835752	0.466307658	-0.83909963	-0.186395987	-1.278066143
30	0.57735027	3	0	-1	0.577350269	-0.57735027	0	-1
35	0.70020754	3	-1.00871191	-0.764564105	0.700207538	-0.36397023	0.168118652	-0.849355749
40	0.83909963	3	-1.98831795	-0.443867713	0.839099631	-0.17632698	0.331386325	-0.773318403
45	1	3	-3	-3.33067E-16	1	-7.4015E-17	0.5	-0.75
50	1.19175359	3	-4.10424172	0.630414938	1.191753593	0.176326981	0.684040287	-0.773318403
55	1.42814801	3	-5.37635472	1.559410094	1.428148007	0.363970234	0.896059121	-0.849355749
60	1.73205081	3	-6.92820323	3	1.732050808	0.577350269	1.154700538	-1
65	2.14450692	3	-8.95081966	5.398364898	2.144506921	0.839099631	1.491803276	-1.278066143
70	2.74747742	3	-11.817693	9.822948256	2.747477419	1.191753593	1.969615506	-1.815207469
75	3.73205081	3	-16.3923048	19.39230485	3.732050808	1.732050808	2.732050808	-3
80	5.67128182	3	-25.2562777	46.74515622	5.67128182	2.747477419	4.20937962	-6.411474128
83.0175	8.16496581	3	-36.5586344	98.4999999	8.164965807	4.02124566	6.093105733	-12.87781249
85	11.4300523	3	-51.3040024	194.4691435	11.4300523	5.67128182	8.550667061	-24.87257811

2) For x = −12 and $\tan 65^\circ = 2.144506921$:

$$ax^2 + bx + c = 3(-12)^2 + (-8.950819655)(-12) + 5.398364898$$
$$= 544.8082008$$
$$= y$$

Figure 15. Eighteen Identical Curves Belonging to the Family $3x^2 = y$.



For *Table 19*, again, values of x range from −15 to +15 in increments of one unit. Based upon page limitations, two pages are necessary to portray this range. Furthermore, the first set of pages lists the tan 5° to tan 45° values, while the second set covers tan 50° thru 85°, inclusively.

Table 19. Tabulation for *Figure 15*.

x plot values	$ax^2 + bx + c = y$ plot values								
	tan 5°	tan 10°	tan 15°	tan 20°	tan 25°	tan 30°	tan 35°	tan 40°	tan 45°
15	924.782173	789.2484065	739.4922678	710.9489626	690.601803	674	659.1047572	644.731363	630
14	821.030794	694.5349552	648.099963	621.4656125	602.4834271	587	573.1134691	559.719681	546
13	723.279415	605.8215039	562.7076581	537.9822625	520.3650512	506	493.122181	480.7079989	468
12	631.528035	523.1080526	483.3153533	460.4989124	444.2466753	431	419.130893	407.6963169	396
11	545.776656	446.3946013	409.9230485	389.0155623	374.1282994	362	351.1396049	340.6846348	330
10	466.025276	375.68115	342.5307436	323.5322122	310.0099234	299	289.1483168	279.6729528	270
9	392.273897	310.9676987	281.1384388	264.0488622	251.8915475	242	233.1570287	224.6612707	216
8	324.522517	252.2542473	225.7461339	210.5655121	199.7731716	191	183.1657406	175.6495887	168
7	262.771138	199.540796	176.3538291	163.082162	153.6547957	146	139.1744525	132.6379066	126
6	207.019758	152.8273447	132.9615242	121.5988119	113.5364198	107	101.1831644	95.62622458	90
5	157.268379	112.1138934	95.56921938	86.11546187	79.41804384	74	69.19187634	64.61454253	60
4	113.516999	77.40044207	64.17691454	56.6321118	51.29966792	47	43.20058825	39.60286048	36
3	75.7656198	48.68699075	38.78460969	33.14876172	29.18129201	26	23.20930016	20.59117843	18
2	44.0142403	25.97353944	19.39230485	15.66541165	13.06291609	11	9.218012071	7.579496385	6
1	18.2628609	9.260088122	6	4.182061572	2.944540167	2	1.22672398	0.567814336	0
0	-1.4885186	-1.453363194	-1.39230485	-1.301288503	-1.173835752	-1	-0.764564105	-0.443867713	0
-1	-15.239898	-6.16681451	-2.78460969	-0.784638578	0.707788329	2	3.244147807	4.544450238	6
-2	-22.991278	-4.880265826	1.823085465	5.732011347	8.58941241	11	13.25285972	15.53276819	18
-3	-24.742657	2.406282858	12.43078062	18.24866127	22.47103649	26	29.26157163	32.52108614	36
-4	-20.494036	15.69283154	29.03847578	36.7653112	42.35266057	47	51.27028354	55.50940409	60
-5	-10.245416	34.97938023	51.64617093	61.28196112	68.23428465	74	79.27899546	84.49772204	90
-6	6.00320458	60.26592891	80.25386609	91.79861105	100.1159087	107	113.2877074	119.48604	126
-7	28.2518251	91.55247759	114.8615612	128.315261	137.9975328	146	153.2964193	160.4743579	168
-8	56.5004456	128.8390263	155.4692564	170.8319109	181.8791569	191	199.3051312	207.4626759	216
-9	9.7490662	172.125575	202.0769516	219.3485608	231.760781	242	251.3138431	260.4509938	270
-10	130.997687	221.4121236	254.6846467	273.8652107	287.6424051	299	309.322555	319.4393118	330
-11	177.246307	276.6986723	313.2923419	334.3818607	349.5240291	362	373.3312669	384.4276297	396
-12	229.494928	337.985221	377.900037	400.8985106	417.4056532	431	443.3399788	455.4159477	468
-13	287.743548	405.2717697	448.5077322	473.4151605	491.2872773	506	519.3486908	532.4042657	546
-14	351.992169	478.5583184	525.1154273	551.9318104	571.1689014	587	601.3574027	615.3925836	630
-15	422.240789	557.8448671	607.7231225	636.4484604	657.0505255	674	689.3661146	704.3809016	720

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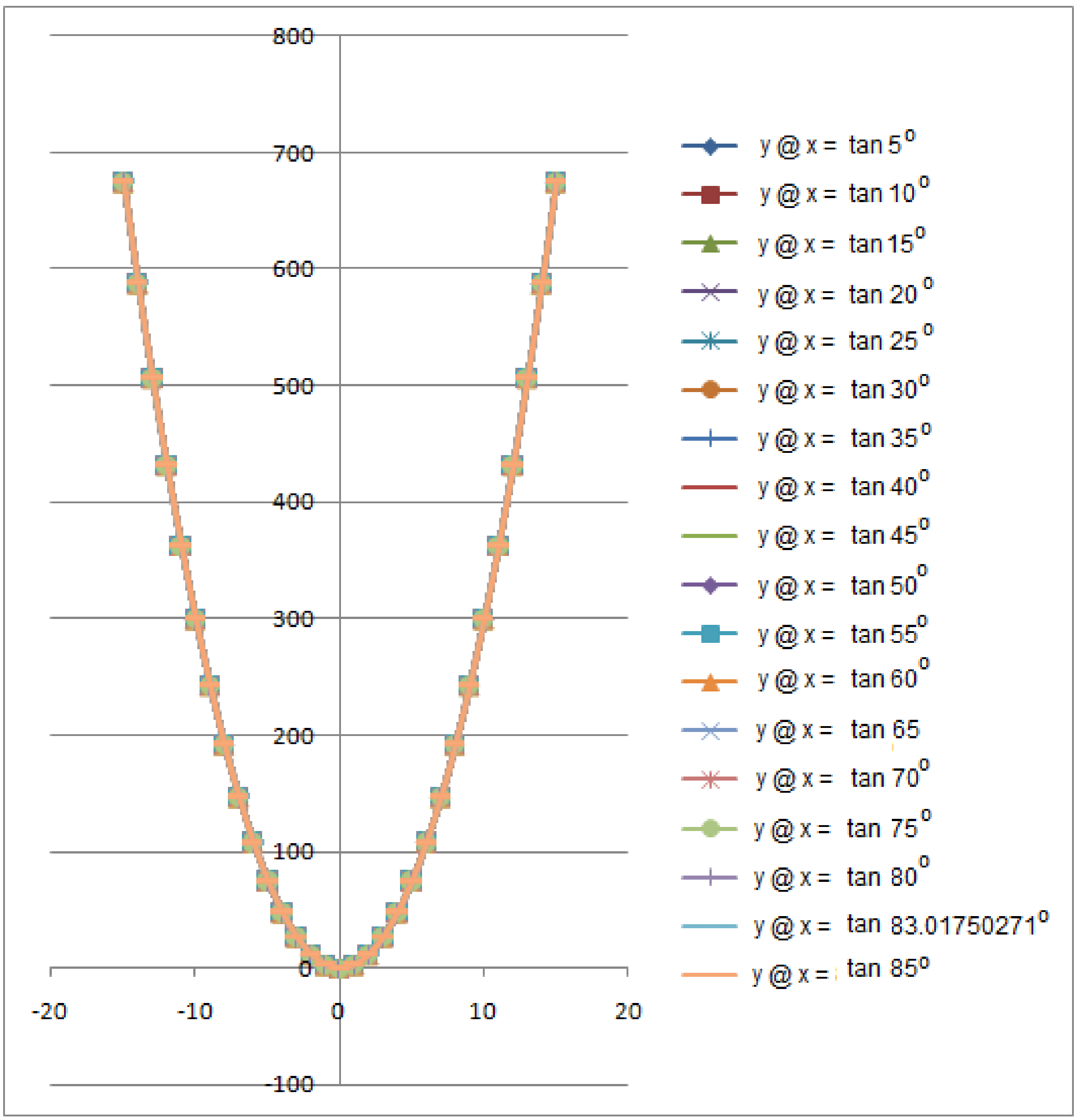
x plot values	$ax^2 + bx + c = y$ **y plot values**								
	tan 50°	tan 55°	tan 60°	tan 65°	tan 70°	tan 75°	tan 80°	tan 83.01750271°	tan 85°
15	614.0667891	595.9140892	574.0769516	546.1360701	507.5575527	448.5077321	342.9009904	224.8204839	99.90910795
14	531.1710309	514.290444	494.0051548	468.0868897	432.3752457	377.900037	281.1572681	174.3791183	64.21311032
13	454.2752726	438.6667987	419.933358	396.0377094	363.1929387	313.2923418	225.4135459	129.9377527	34.51711269
12	383.3795143	369.0431534	351.8615612	329.988529	300.0106318	254.6846467	175.6698236	91.49638714	10.82111506
11	318.483756	305.4195081	289.7897645	269.9393487	242.8283248	202.0769515	131.9261013	59.05502154	-6.87488257
10	259.5879977	247.7958629	233.7179677	215.8901683	191.6460179	155.4692564	94.18237902	32.61365594	-18.5708802
9	206.6922395	196.1722176	183.6461709	167.840988	146.4637109	114.8615612	62.43865674	12.17229034	-24.26687783
8	159.7964812	150.5485723	139.5743742	125.7918077	107.2814039	80.25386605	36.69493446	-2.26907526	-23.96287546
7	118.9007229	110.924927	101.5025774	89.74262731	74.09909698	51.6461709	16.95121218	-10.71044086	-17.65887309
6	84.00496462	77.30128176	69.43078062	59.69344697	46.91679002	29.03847575	3.2074899	-13.15180646	-5.35487072
5	55.10920634	49.67763648	43.35898385	35.64426662	25.73448306	12.4307806	-4.53623238	-9.59317206	12.94913165
4	32.21344806	28.0539912	23.28718708	17.59508628	10.5521761	1.82308545	-6.27995466	-0.03453766	37.25313402
3	15.31768978	12.43034593	9.21539031	5.545905933	1.369869136	-2.7846097	-2.02367694	15.52409674	67.55713639
2	4.421931498	2.806700648	1.14359354	-0.503274412	-1.812437824	-1.39230485	8.23260078	37.08273114	103.8611388
1	-0.473826782	-0.816944629	-0.92820323	-0.552454757	1.005255216	6	24.4888785	64.64136554	146.1651411
0	0.630414938	1.559410094	3	5.398364898	9.822948256	19.39230485	46.74515622	98.19999994	194.4691435
-1	7.734656658	9.935764817	12.92820323	17.34918455	24.6406413	38.7846097	75.00143394	137.7586343	248.7731459
-2	20.83889838	24.31211954	28.85640646	35.30000421	45.45833434	64.17691455	109.2577117	183.3172687	309.0771482
-3	39.9431401	44.68847426	50.78460969	59.25082386	72.27602738	95.5692194	149.5139894	234.8759031	375.3811506
-4	65.04738182	71.06482899	78.71281292	89.20164352	105.0937204	132.9615243	195.7702671	292.4345375	447.685153
-5	96.15162354	103.4411837	112.6410162	125.1524632	143.9114135	176.3538291	248.0265448	355.9931719	525.9891554
-6	133.2558653	141.8175384	152.5692194	167.1032828	188.7291065	225.746134	306.2828225	425.5518063	610.2931577
-7	176.360107	186.1938932	198.4974226	215.0541025	239.5467995	281.1384388	370.5391003	501.1104407	700.5971601
-8	225.4643487	236.5702479	250.4256258	269.0049221	296.3644926	342.5307437	440.795378	582.6690751	796.9011625
-9	280.5685904	292.9466026	308.3538291	328.9557418	359.1821856	409.9230485	517.0516557	670.2277095	899.2051648
-10	341.6728321	355.3229573	372.2820323	394.9065614	427.9998787	483.3153534	599.3079334	763.7863439	1007.509167
-11	408.7770739	423.699312	442.2102355	466.8573811	502.8175717	562.7076582	687.5642111	863.3449783	1121.81317
-12	481.8813156	498.0756668	518.1384388	544.8082008	583.6352647	648.0999631	781.8204889	968.9036127	1242.117172
-13	560.9855573	578.4520215	600.066642	628.7590204	670.4529578	739.4922679	882.0767666	1080.462247	1368.421174
-14	646.089799	664.8283762	687.9948452	718.7098401	763.2706508	836.8845728	988.3330443	1198.020882	1500.725177
-15	737.1940407	757.2047309	781.9230485	814.6606597	862.0883439	940.2768776	1100.589322	1321.579516	1639.029179

Notice that the first *y entry* given in *Table 19* displays a value of 924.782173, just as computed above. Secondly, *Table 19* returns a *y entry* value of 544.8082008 for the condition when x = −12 and $\tan 65^{o}$; just as determined above.

The intention of *Figure 16* is to show that ***all*** *curves* displayed in *Figure 15 are identical* by virtue of the fact that each curve each can be translated to a new location such that it entirely superimposes upon or overlaps the others.

Figure 16. The Collapsing of Eighteen Curves Represented in *Figure 15*.



This is demonstrated through a *step function* derived as follows:

$$ax^2 + [(bx - bx) + b(x_M - x_M) + a(x_M{}^2 - x_M{}^2) + (c - c)] = y$$
$$ax^2 + (bx + 2ax_M x) + b(x_M - x_M) + a(x_M{}^2 - x_M{}^2) + (c - c) = y$$
$$a(x^2 + 2x_M x + x_M{}^2) + b(x + x_M) + c - (ax_M{}^2 + bx_M + c) = y$$
$$a(x + x_M)^2 + b(x + x_M) + (c - y_M) = y$$

Such derivation indicates that the *Parabolic Curve* $ax^2 = y$ can be <u>precisely mapped</u> by an alternate *step function* $a(x + x_M)^2 + b(x + x_M) + (c - y_M) = y$, hereinafter termed the ***Normalization Transformation for Parabolic Functions***, which:

- Is comprised of $x' = x + x_M$ terms and:
- Exhibits the *same format* as the *Parabolic Function* $ax'^2 + bx' + c' = y$ where,

  $c' = c - y_M$

Table 20 presents the tabulation for *Figure 16*. Notice that for each row presented, all columns return the same exact respective values for y.

Such overall identity is obtained by applying the *Normalization Transformation for Parabolic Functions* to each of the eighteen curves illustrated *Figure 15*. As such, *Table 20* ordinate entries may be obtained in either one of two ways as listed below:

1) *Table 18* values, may be inserted into the *Normalization Transformation for Parabolic Functions* for each and every specified θ value. Missing entries can be added using the method previously described above. For example, since $\theta = 4^o$ does <u>not</u> appear in *Table 18*, it may be added to it, such that:

   For $\tan 4^o = 0.069926811$,

$$b = a(\frac{1 - 3\tan^2\theta}{2\tan\theta}) = 3[\frac{1 - 3(0.069926811)^2}{2(0.069926811)}] = 21.13632873$$
$$c = \frac{a}{2}(\tan^2\theta - 1) = \frac{3}{2}[(0.069926811)^2 - 1] = -1.492265361$$
$$x_M = -\frac{b}{2a} = -\frac{21.13632873}{2(3)} = -3.522721455$$
$$y_M = ax_M{}^2 + bx_M + c = -38.72096471$$

$$a(x+x_M)^2+b(x+x_M)+(c-y_M)=y$$
$$3(4-3.522721455)^2+21.13632873(4-3.522721455)-(1.492265361-38.72096471)=y$$
$$0.683384429+10.08791622+37.22869935=48$$

2) *Normalization Transformation for Parabolic Functions* entries can be ascertained by adding *calculated values* specified below to respective *Table 19* entries as follows:

Where,

$$\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}=\mathrm{y} \qquad \textit{[Ref. Table 19]}$$
$$a(x+x_M)^2+b(x+x_M)+(c-y_M)=y \qquad \textit{[Ref. Table 20]}$$
$$a(x^2+2x_M x+x_M{}^2)+b(x+x_M)+c-y_M=y$$
$$(ax^2+bx+c)+(2ax_M x+ax_M{}^2+bx_M-y_M)=y$$

Three sets of calculations to determine *Table 20* ordinate values are shown below based upon *Table 19* entries:

- For the *Table 19* entry x = 15 and $\tan 5^{\circ}$,

$$(ax^2+bx+c)+(2ax_M x+ax_M{}^2+bx_M-y_M)=y$$
$$924.782173+[2(3)(\text{-}2.791896578)(15)+3(\text{-}2.791896578)^2+16.75137947(\text{-}2.791896578)+24.87257811]=y$$
$$924.782173+(-251.270692+23.38405951-46.76811902+24.87257811)=y$$
$$924.782173-249.782173=675$$

- For the *Table 19* entry x = 15 and $\tan 15^{\circ}$,

$$(ax^2+bx+c)+(2ax_M x+ax_M{}^2+bx_M-y_M)=y$$
$$739.4922678+[2(3)(\text{-}0.732050808)(15)+3(\text{-}0.732050808)^2+4.392304845(\text{-}0.732050808)+3]=y$$
$$739.4922678+(-65.8845727+1.607695156-3.215390311+3)=y$$
$$739.4922678-64.4922678=675$$

- For the *Table 19* entry x = -12 and $\tan 65^{\circ}$,

$$(ax^2+bx+c)+(2ax_M x+ax_M{}^2+bx_M-y_M)=y$$
$$544.8082008+[2(3)(1.491803276)(\text{-}12)+3(1.491803276)^2\ \text{-}8.95081966(1.491803276)+1.278066143]=y$$
$$544.8082008-112.8082008=432$$

Table 20. Tabulation for *Figure 16*.

X plot values	$a(x+x_M)^2+b(x+x_M)+c-y_M=y$ Plot Values for the Following Curves																	
	tan 5°	tan 10°	tan 15°	tan 20°	tan 25°	tan 30°	tan 35°	tan 40°	tan 45°	tan 50°	tan 55°	tan 60°	tan 65°	tan 70°	tan 75°	tan 80°	tan 83.01750271°	tan 85°
15	675	675	675	675	675	675	675	675	675	675	675	675	675	675	675	675	675	675
14	588	588	588	588	588	588	588	588	588	588	588	588	588	588	588	588	588	588
13	507	507	507	507	507	507	507	507	507	507	507	507	507	507	507	507	507	507
12	432	432	432	432	432	432	432	432	432	432	432	432	432	432	432	432	432	432
11	363	363	363	363	363	363	363	363	363	363	363	363	363	363	363	363	363	363
10	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300
9	243	243	243	243	243	243	243	243	243	243	243	243	243	243	243	243	243	243
8	192	192	192	192	192	192	192	192	192	192	192	192	192	192	192	192	192	192
7	147	147	147	147	147	147	147	147	147	147	147	147	147	147	147	147	147	147
6	108	108	108	108	108	108	108	108	108	108	108	108	108	108	108	108	108	108
5	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
4	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48
3	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27
2	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
-2	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
-3	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27
-4	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48
-5	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75	75
-6	108	108	108	108	108	108	108	108	108	108	108	108	108	108	108	108	108	108
-7	147	147	147	147	147	147	147	147	147	147	147	147	147	147	147	147	147	147
-8	192	192	192	192	192	192	192	192	192	192	192	192	192	192	192	192	192	192
-9	243	243	243	243	243	243	243	243	243	243	243	243	243	243	243	243	243	243
-10	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300
-11	363	363	363	363	363	363	363	363	363	363	363	363	363	363	363	363	363	363
-12	432	432	432	432	432	432	432	432	432	432	432	432	432	432	432	432	432	432
-13	507	507	507	507	507	507	507	507	507	507	507	507	507	507	507	507	507	507
-14	588	588	588	588	588	588	588	588	588	588	588	588	588	588	588	588	588	588
-15	675	675	675	675	675	675	675	675	675	675	675	675	675	675	675	675	675	675

TRUE SCANS

Algorithmic relationships expressed in *Table 18* are applied in the following example, where:

a) *It is desired to plot a Parabolic Curve of the form* $3x^2 + bx + c = y$ *which is identical in shape to the curve* $3x^2 - 200 = y$, *possesses its very same* x_1 *root, and has low point coordinates as follows:*

$$x_M = 6.093105735$$

$$y_M = \text{-}12.8778125$$

b) *It is desired to plot a second Parabolic Curve of the form* $ax^2 + bx + c = y$ *which is identical in shape to the curve* $3x^2 - 200 = y$, *possesses its very same* x_1 *root, but exhibits another low point that is different than* $x_M = 6.093105735$ and $y_M = \text{-}12.8778125$

Solution:

a) The *roots* of the *given curve* $3x^2 - 200 = y$ *are* determined at $y = 0$ as,

$$3x^2 - 200 = 0$$

$$\bar{x} = x_1; x_2 = \pm\sqrt{\frac{200}{3}}$$

$$= \pm 8.164965809$$

For the *desired equation* $ax^2 + b_1 x + c_1 = y$

$$x_{M-1} = -\frac{b_1}{2a} = 6.093105735$$

$$b_1 = -2(\text{a})6.093105735$$

$$= -2(3)6.093105735$$

$$= -36.55863441$$

$$y_{M-1} = ax_{M-1}{}^2 + b_1 x_{M-1} + c_1 = \text{-}12.8778125$$

$$c_1 = \text{-}12.8778125 - (ax_{M-1}{}^2 + b_1 x_{M-1})$$

$$= -12.8778125 - [\,3(6.093105735)^2 \ \text{-}\, 36.55863441(6.093105735)\,]$$

$$= -12.8778125 - 111.3778125 + 222.755625$$

$$= 98.5$$

Results:

- $3x^2 - 200 = y$
- $3x^2 \ \text{-}\, 36.55863441\text{x} + 98.5 = y$

TRUE SCANS

b) Now letting:

$$x_1 = \tan\theta = +\sqrt{\frac{200}{3}} = +8.164965809 = \tan 83.01750271^o$$

$$x_2 = -\tan(2\theta) = -\tan 2(83.01750271)^o$$
$$= -\tan 166.0350054^o$$
$$= 0.248679161$$

Then for the *second desired equation* $ax^2 + b_2x + c_2 = y$

$$\frac{c_2}{a} = x_1x_2$$ *(Ref. earlier table Section 14.1.2)*

$$c_2 = ax_1x_2$$
$$= 3\sqrt{\frac{200}{3}}(0.248679161)$$
$$= 6.091370558$$

$$\frac{b_2}{a} = \frac{\tan\theta + \tan^3\theta}{1-\tan^2\theta}$$ *(Ref. later table Section 14.1.2)*

$$b_2 = 3\left(\frac{\sqrt{\frac{200}{3}} + \frac{200}{3}\sqrt{\frac{200}{3}}}{1-\frac{200}{3}}\right)$$
$$= 3\sqrt{\frac{200}{3}}\left(\frac{3+200}{3-200}\right)$$
$$= -3\sqrt{\frac{200}{3}}\left(\frac{203}{197}\right)$$
$$= -25.24093491$$

Then,

$$3x^2 - 25.24093491x + 6.091370558 = y$$

For this equation:

$$x_{M-2} = -\frac{b_2}{2a} = \frac{25.24093491}{6} = 4.206822486$$

$$y_{M-2} = a(x_{M-2})^2 + b_2x_{M-2} + c_2 = 3(4.206822486)^2 - 25.24093491(4.206822486) + 6.091370558$$
$$= -47.0006957$$

Figure 17 charts these curves in relationship to their $3x^2 = y$ *Parent Function*. Notice the *low point* of the *Parent Function* lies directly upon the origin. Furthermore, the *three other curves* itemized above intersect at a shared root value of x_1 equals +8.164965809. The curve $3x^2 - 200 = y$ resides directly below the *Parent Function* by a displacement of 200 units.

Figure 18 demonstrates the four curves to be of identical shape because the three resulting curves collapse upon their *Parent Curve* once the proper *translation relationships* are applied. This is easily confirmed, therein, since respective equation y values are exactly the same for each and every entry of x.

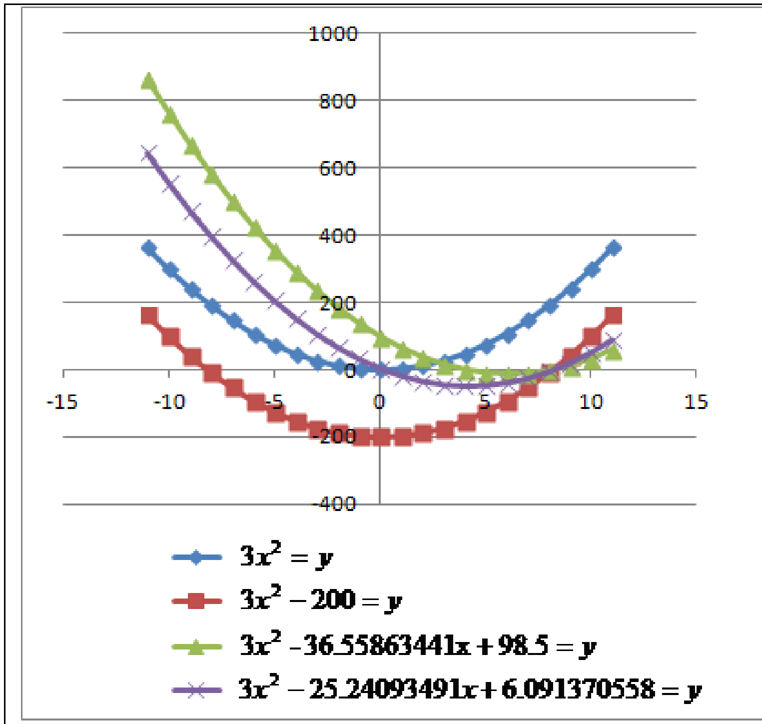
In this case, an example calculation for the following two equation when x equals a *minus five* is given below:

$$3(x + x_{M-1})^2 + b_1(x + x_{M-1}) + c_1 - y_{M-1} = y$$
$$3(-5 + 6.093105735)^2 + b_1(-5 + 6.093105735) + 98.5 + 12.8778125 = y$$
$$3(1.093105735)^2 \text{ - } 36.55863441(1.093105735) + 98.5 + 12.8778125 = y$$
$$3.584640444 \text{ - } 39.96245294 + 98.5 + 12.8778125 = y$$
$$75 = y$$

$$3(x + x_{M-2})^2 + b_2(x + x_{M-2}) + c_1 - y_{M-2} = y$$
$$3(-5 + 4.206822486)^2 + b_2(-5 + 4.206822486) + 6.091370558 + 47.0006957 = y$$
$$3(-0.793177514)^2 \text{ - } 25.24093491(-0.793177514) + 6.091370558 + 47.0006957 = y$$
$$1.887391706 + 20.020542 + 6.091370558 + 47.0006957 = y$$
$$75 = y$$

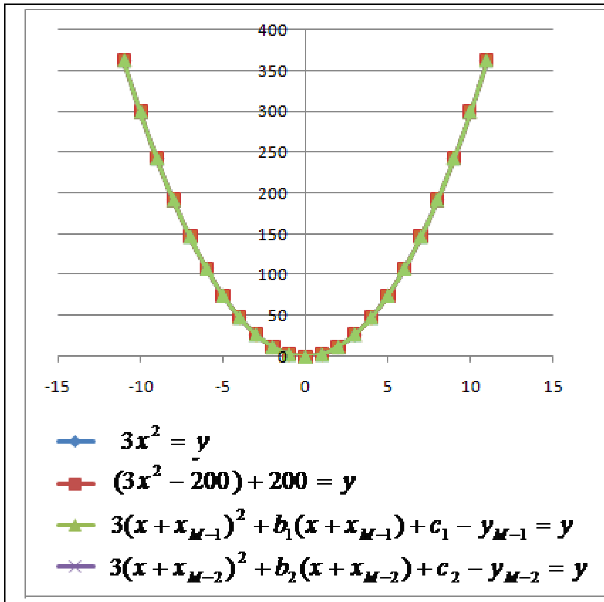
TRUE SCANS

Figure 17. Plot of Four Identical Parabolic Curves with $3x^2 = y$ Low Point Located at Origin.



X plot Values	Y plot Values			
	$3x^2 = y$		$3x^2 - 36.55863441x + 98.5 = y$	$3x^2 - 25.24093491x + 6.091370558 = y$
11	363	163	59.35502149	91.44108654
10	300	100	32.9136559	53.68202145
9	243	43	12.47229031	21.92295636
8	192	-8	-1.96907528	-3.836108732
7	147	-53	-10.41044087	-23.59517382
6	108	-92	-12.85180646	-37.35423891
5	75	-125	-9.29317205	-45.113304
4	48	-152	0.26546236	-46.87236909
3	27	-173	15.82409677	-42.63143418
2	12	-188	37.38273118	-32.39049927
1	3	-197	64.94136559	-16.14956436
0	0	-200	98.5	6.091370548
-1	3	-197	138.0586344	34.33230546
-2	12	-188	183.6172688	68.57324037
-3	27	-173	235.1759032	108.8141753
-4	48	-152	292.7345376	155.0551102
-5	75	-125	356.2931721	207.2960451
-6	108	-92	425.8518065	265.53698
-7	147	-53	501.4104409	329.7779149
-8	192	-8	582.9690753	400.0188498
-9	243	43	670.5277097	476.2597847
-10	300	100	764.0863441	558.5007196
-11	363	163	863.6449785	646.7416546

Figure 18. Plot of Four Identical Parabolic Curves Collapsing upon the Parent $3x^2 = y$ Curve.



Where,

$$x_M = x_{M-1} = 6.093105735$$
$$y_M = y_{M-1} = -12.8778125$$
$$b_1 = -36.55863441$$
$$c_1 = 98.5$$

$$x_{M-2} = 4.206822486$$
$$y_{M-2} = -47.0006957$$
$$b_2 = -25.24093491$$
$$c_2 = 6.091370558$$

X plot Values	Y plot Values			
	$3x^2 = y$	$(3x^2 - 200) + 200 = y$	$3(x + x_{M-1})^2 + b_1(x + x_{M-1}) + c_1 - y_{M-1} = y$	$3(x + x_{M-2})^2 + b_2(x + x_{M-2}) + c_1 - y_{M-2} = y$
11	363	363	363	363
10	300	300	300	300
9	243	243	243	243
8	192	192	192	192
7	147	147	147	147
6	108	108	108	108
5	75	75	75	75
4	48	48	48	48
3	27	27	27	27
2	12	12	12	12
1	3	3	3	3
0	0	0	0	0
-1	3	3	3	3
-2	12	12	12	12
-3	27	27	27	27
-4	48	48	48	48
-5	75	75	75	75
-6	108	108	108	108
-7	147	147	147	147
-8	192	192	192	192
-9	243	243	243	243
-10	300	300	300	300
-11	363	363	363	363

RUN SCANS

In summary, the following steps should be taken to ascertain *coefficients* which belong to a *Parabolic Function* of the form $ax^2 + bx + c = y$ that characterizes any constructed *Parabolic Curve* whose low point resides at a given location relative to any arbitrarily specified origin:

1) First, in order to verify the given curve belongs to the family $ax^2 = y$, a plot of the curve is to be generated with respect to its *low point*.

   As such, each measured y value represents the *vertical measurement* from the *low point* of the curve to a relative elevation, or latitude of any other point which resides upon the curve.

   Likewise, each measured x value then represents the *horizontal distance* measured from the *low point* of the curve to a vertical projection of the same respective other point previously identified above.

   Then such y measurement is to be divided by the square of its respective x measurement in order to determine a value for the coefficient *a*. If the results for all sets of x and y values are the same, then the curve is *Parabolic*.

2) From x_M, the horizontal distance measured from any arbitrarily specified origin to the vertical projection of the low point on the given curve, the coefficient *b* is calculated as follows:

   $$b = -2ax_M$$

3) From y_M, the vertical distance measured from the arbitrarily specified origin to the horizontal projection of low point on the given curve, the coefficient *c* is calculated as follows:

   $$c = y_M - ax_M{}^2 - bx_M$$

TRUE SCANS

A simple example for this approach follows:

It is desired to find the $ax^2 + bx + c = y$ *Parabolic Function for a curve whose locus of points is represented as follows for an origin which is located 14 units to the right and 12 units above the given curve's low point:*

x	y	$a_{CALC} = y/x^2$
1.2	25.2	17.5
2.5	109.375	17.5
2.5	109.375	17.5
4.4	338.8	17.5
5	437.5	17.5

Solution:

1) Since all of the values in the right column equal 17.5, then the given graph identifies *parabolic coordinates*.
2) $b = -2ax_M = -2(17.5)(-14) = 490$
3) $c = y_M - ax_M{}^2 - bx_M = -12 - (17.5)(-14)^2 - 490(-14) = 3{,}418$

Or,

$$17.5x^2 + 490x + 3{,}418 = y$$

14.2. Generalized Cubic Curve Mapping.

Figure 19 shows a typical *Generalized Cubic Curve*. It may be represented either as a sketch, or a plot of points on a graph since the latter may be developed from the former; and vice versa. However, when represented as a plot of points on a graph, *plot values* are dependent upon where the origin is placed at. The notations z_R, z_S, and z_T denote respective *roots* for any arbitrarily selected origin.

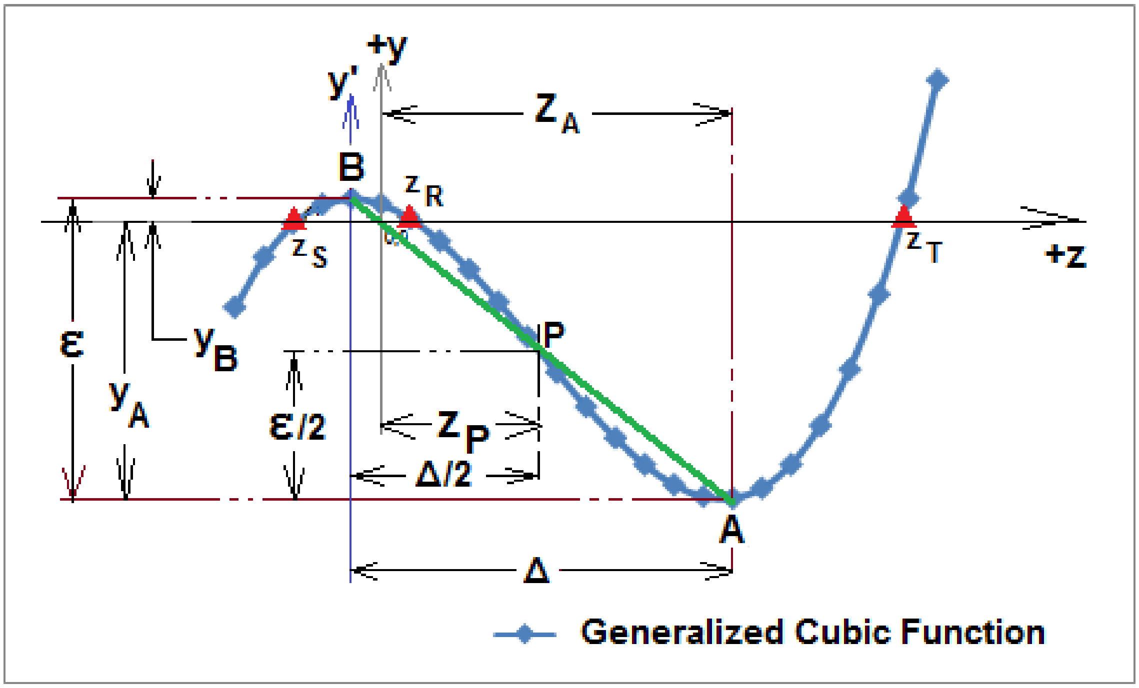
More specifically, *Figure 19* applies only to curve families which satisfy all constraints imposed upon them by the *Generalized Cubic Function (Ref. Equation 32)*:

$$z^3 + \frac{\beta}{\alpha}z^2 + \frac{\gamma}{\alpha}z + \frac{\delta}{\alpha} = \frac{y}{\alpha}$$

Or,

$$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$$

Figure 19. The Generalized Cubic Function.



14.2.1. Singularity Proof.

The proof presented below is <u>converse</u> to that presented in *Section 14.1.1* since it moves from the *Generalized Cubic Function*, or *family* curve, to its *Parent Curve*, $z'^3 + \sigma z'^2 = y'$.

So, where:

$$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$$

The first derivative of the curve is:

$$3z^2 dz + (2\beta' z)dz + \gamma' dz = dy$$

This applies to identifiable locations 'A' and 'B' on the curve presented in *Figure 19* where the slope equals zero, as follows:

$$3z^2 + 2\beta' z + \gamma' = dy/dz = 0$$

After multiplying thru by a factor of 1/3, rearranging terms, and finally *Completing the Square*, this equates to:

$$z^2 + \frac{2}{3}\beta' z = -\frac{\gamma'}{3}$$

$$z^2 + \frac{2}{3}\beta' z + (\frac{1}{3}\beta')^2 = -\frac{\gamma'}{3}(\frac{3}{3}) + (\frac{1}{3}\beta')^2$$

$$(z + \frac{1}{3}\beta')^2 = -\frac{3\gamma'}{9} + \frac{\beta'^2}{9}$$

$$z_A, z_B = \frac{1}{3}[-\beta' \pm \sqrt{\beta'^2 - 3\gamma'}]$$

For the *Family Function* noted above which possesses roots of z_R, z_S, and z_T:

$$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$$

$$(z - z_R)(z - z_S)(z - z_T) = y_{TRANSFORMED}$$

$$(z - z_R)[z^2 - (z_S + z_T)z + z_S z_T] = y_{TRANSFORMED}$$

$$z^3 - (z_R + z_S + z_T)z^2 + (z_R z_S + z_R z_T + z_S z_T)z - z_R z_S z_T = y_{TRANSFORMED}$$

By comparing respective terms, the following identities are arrived at:

- $\beta' = -(z_R + z_S + z_T)$
- $\gamma' = z_R z_S + z_R z_T + z_S z_T$
- $\delta' = -z_R z_S z_T$

Now consider a new, *transformed* y-axis (ref. *Figure 19*) hereinafter designated as y′ which vertically aligns with Point B such that horizontal offsets to the curve's *existing roots* are relegated as:

- $z_R' = z_R - z_B$
- $z_S' = z_S - z_B$
- $z_T' = z_T - z_B$

Accordingly, the *transformed Cubic Function* which employs such root structure is derived as follows:

$$(z' - z_R')(z' - z_S')(z' - z_T') = y'$$

$$(z' - z_R')[z'^2 - (z_S' + z_T')z' + z_S' z_T'] = y'$$

$$z'^3 - (z_R' + z_S' + z_T')z'^2 + (z_R' z_S' + z_R' z_T' + z_S' z_T')z' - z_R' z_S' z_T' = y'$$

$$z'^3 + \sigma z'^2 + \tau z + \nu = y'$$

Such that,

$$\sigma = -(z_R' + z_S' + z_T')$$

$$= -(z_R - z_B + z_S - z_B + z_T - z_B$$

$$= -(z_R + z_S + z_T - 3z_B)$$

$$= \beta' + 3z_B$$

$$= \beta' + 3[\frac{1}{3}(-\beta' - \sqrt{\beta'^2 - 3\gamma'})]$$

$$= \beta' - \beta' - \sqrt{\beta'^2 - 3\gamma'})$$

$$= -\sqrt{\beta'^2 - 3\gamma'}$$

$$\tau = z_R' z_S' + z_R' z_T' + z_S' z_T'$$

$$= (z_R - z_B)(z_S - z_B) + (z_R - z_B)(z_T - z_B) + (z_S - z_B)(z_T - z_B)$$
$$= (z_R z_S + z_R z_T + z_S z_T) - z_B(z_R + z_S + z_R + z_T + z_S + z_T) + 3z_B{}^2$$
$$= (z_R z_S + z_R z_T + z_S z_T) - 2z_B(z_R + z_S + z_T) + 3z_B{}^2$$
$$= \gamma' + 2\beta' z_B + 3z_B{}^2$$
$$= \gamma' + 2\beta'[\frac{1}{3}(-\beta' - \sqrt{\beta'^2 - 3\gamma'})] + 3[\frac{1}{3}(-\beta' - \sqrt{\beta'^2 - 3\gamma'})]^2$$
$$= \gamma' + \frac{2}{3}(-\beta'^2 - \beta\sqrt{\beta'^2 - 3\gamma'}) + \frac{1}{3}(\beta'^2 + 2\beta'\sqrt{\beta'^2 - 3\gamma'}) + \beta'^2 - 3\gamma')$$
$$= \gamma' + \frac{2}{3}(-\beta'^2 - \beta\sqrt{\beta'^2 - 3\gamma'}) + \frac{2}{3}\beta'^2 + \frac{2}{3}\beta'\sqrt{\beta'^2 - 3\gamma'}) - \gamma'$$
$$= \frac{2}{3}(-\beta'^2 - \beta\sqrt{\beta'^2 - 3\gamma'}) + \frac{2}{3}\beta'^2 + \frac{2}{3}\beta'\sqrt{\beta'^2 - 3\gamma'})$$
$$= -\frac{2}{3}\beta\sqrt{\beta'^2 - 3\gamma'}) + \frac{2}{3}\beta'\sqrt{\beta'^2 - 3\gamma'})$$
$$= 0$$

$$\nu = -z_R' z_S' z_T'$$
$$= -(z_R - z_B)(z_S - z_B)(z_T - z_B)$$
$$= -(z_R - z_B)[z_S z_T - (z_S + z_T)z_B + z_B{}^2]$$
$$= -z_R z_S z_T + (z_R z_S + z_R z_T + z_S z_T)z_B - (z_R + z_S + z_T)z_B{}^2 + z_B{}^3$$
$$= \delta' + \gamma' z_B + \beta' z_B{}^2 + z_B{}^3$$

Notice that ν represents the *function* for *Equation 32* once divided thru by α.

Moreover,

$$\nu = \delta' + \gamma' z_B + \beta' z_B{}^2 + z_B{}^3$$
$$= \delta' + \gamma'[\frac{1}{3}(-\beta' - \sqrt{\beta'^2 - 3\gamma'})] + \beta'[\frac{1}{3}(-\beta' - \sqrt{\beta'^2 - 3\gamma'})]^2 + [\frac{1}{3}(-\beta' - \sqrt{\beta'^2 - 3\gamma'})]^3$$

Where,

$$z_B = \frac{9}{27}[-\beta' \pm \sqrt{\beta'^2 - 3\gamma'}]$$

$$z_B{}^2 = [\frac{1}{3}(-\beta' - \sqrt{\beta'^2 - 3\gamma'})]^2$$
$$= \frac{3}{27}[\beta'^2 + 2\beta'\sqrt{\beta'^2 - 3\gamma'} + \beta'^2 - 3\gamma']$$
$$= \frac{3}{27}[2\beta'^2 + 2\beta'\sqrt{\beta'^2 - 3\gamma'} - 3\gamma']$$

$$z_B{}^3 = [\frac{1}{3}(-\beta' - \sqrt{\beta'^2 - 3\gamma'})]^3$$

$$= \frac{1}{27}[-\beta'-\sqrt{\beta'^2-3\gamma'}][2\beta'^2+2\beta'\sqrt{\beta'^2-3\gamma'}-3\gamma']$$

$$= \frac{1}{27}[-2\beta'^3-2\beta'^2\sqrt{\beta'^2-3\gamma'}+3\beta'\gamma'-2\beta'^2\sqrt{\beta'^2-3\gamma'}-2\beta'(\beta'^2-3\gamma')+3\gamma'\sqrt{\beta'^2-3\gamma'}]$$

$$= \frac{1}{27}[-2\beta'^3-2\beta'^2\sqrt{\beta'^2-3\gamma'}+3\beta'\gamma'-2\beta'^2\sqrt{\beta'^2-3\gamma'}-2\beta'^3+6\beta'\gamma'+3\gamma'\sqrt{\beta'^2-3\gamma'}]$$

$$= \frac{1}{27}[-4\beta'^3+(3\gamma'-4\beta'^2)\sqrt{\beta'^2-3\gamma'}+9\beta'\gamma']$$

Then,

$$\nu = \delta'+\gamma' z_B + \beta' z_B{}^2 + z_B{}^3$$

$$= \frac{1}{27}[27\delta'+9\gamma'(-\beta'-\sqrt{\beta'^2-3\gamma'})+3\beta'(2\beta'^2+2\beta'\sqrt{\beta'^2-3\gamma'}-3\gamma')-4\beta'^3+(3\gamma'-4\beta'^2)\sqrt{\beta'^2-3\gamma'}+9\beta'\gamma']$$

$$= \frac{1}{27}[27\delta'-9\gamma'\sqrt{\beta'^2-3\gamma'}+6\beta'^3+6\beta'^2\sqrt{\beta'^2-3\gamma'}-9\beta'\gamma'-4\beta'^3+(3\gamma'-4\beta'^2)\sqrt{\beta'^2-3\gamma'}]$$

$$= \frac{1}{27}[2\beta'^3+(2\beta'^2-6\gamma')\sqrt{\beta'^2-3\gamma'}-9\beta'\gamma'+27\delta']$$

Therefore,

$$z'^3+\sigma z'^2+\tau z'+\nu = y'$$

$$z'^3+\sigma z'^2+\nu = y'$$

$$z'^3-\sqrt{\beta'^2-3\gamma'}\,z'^2+\frac{1}{27}[27\delta'-9\beta'\gamma'+(2\beta'^2-6\gamma')\sqrt{\beta'^2-3\gamma'}+2\beta'^3] = y'$$

In order to prove *singularity* for the following two curves, it becomes necessary to validate that they're identical.

- $z^3+\beta' z^2+\gamma' z+\delta' = y_{TRANSFORMED}$
- $z'^3+\sigma z'^2+\nu = y'$

This is accomplished by showing that these *two curves* <u>occupy</u> *identical points* throughout their respective spans.

Since both functions itemized above are *general*, such validation should apply to virtually any given *cubic function*.

Again viewing *Figure 19*, it is easily verified that the roots z_R, z_S, and z_T apply to an origin which resides at the intersection of the z-axis and y-axis. All intermediate points on the curve satisfy the equation:

$$z^3+\beta' z^2+\gamma' z+\delta' = y_{TRANSFORMED}\,.$$

Now, if this same curve were to be viewed from a new, translated origin which is placed at the intersection of the z-axis, and the y'-axis, examination of *Figure 19* discloses that for any and all $y_{TRANSFORMED}$ values, a distinct value of z′ exists which is equal to the following:

$$z' = z - z_B$$

Where, z_B is considered to be negative.

This viewpoint is reiterated via the following graph which gives a plot of the *same curve*, but from different relative origins:

Table 21. Curve Plot Relative to Two Different Origins.

z Values	z′ Values	$y_{TRANSFORMED}$ = y′ Values
z_1	$z_1 - z_B$	y_1
z_2	$z_2 - z_B$	y_2
z_3	$z_3 - z_B$	y_3
z_4	$z_4 - z_B$	y_4
z_5	$z_5 - z_B$	y_5

Notice that *Table 21* maps the same exact curve from two different origins, or perspectives, which are horizontally offset from each other by a distance of z_B. Since each horizontal offset is the same for every value of $y_{TRANSFORMED}$ = y′, the two plots must fall, or collapse, onto a singular curve!

Now, the bottom of the two equations rendered above transforms as follows:

$$\begin{aligned} y' = y_{TRANSFORMED} &= z'^3 + \sigma(z')^2 + \nu \\ &= (z - z_B)^3 + \sigma(z - z_B)^2 + \nu \\ &= z^3 - (3z^2)z_B + (3z)z_B{}^2 - z_B{}^3 + \sigma[z^2 - (2z)z_B + z_B{}^2] + \nu \\ &= z^3 + \beta' z^2 + \gamma' z + \delta' \qquad \textit{(See Above)} \end{aligned}$$

Then,

$$z^3 - (3z^2)z_B + (3z)z_B{}^2 - z_B{}^3 + \sigma[z^2 - (2z)z_B + z_B{}^2] + \nu = z^3 + \beta' z^2 + \gamma' z + \delta'$$

$$-(3z^2)z_B + (3z)z_B{}^2 - z_B{}^3 + \sigma[z^2 - (2z)z_B + z_B{}^2] + \nu = +\beta' z^2 + \gamma' z + \delta'$$

Now, where:

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$

$$\begin{aligned} \nu &= \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta'] \\ &= \frac{1}{27}[2\beta'^3 - (2\beta'^2 - 6\gamma')\sigma - 9\beta'\gamma' + 27\delta'] \end{aligned}$$

The aforementioned equation reduces to an identity as follows:

$$-(3z^2)z_B+(3z)z_B{}^2-z_B{}^3+\sigma[z^2-(2z)z_B+z_B{}^2]+\nu=+\beta' z^2+\gamma' z+\delta'$$

$$-(3z^2)z_B+(3z)z_B{}^2-z_B{}^3+\sigma[z^2-(2z)z_B+z_B{}^2]+\frac{1}{27}[2\beta'^3-(2\beta'^2-6\gamma')\sigma-9\beta'\gamma'+27\delta']=+\beta' z^2+\gamma' z+\delta'$$

$$-(3z^2)z_B+(3z)z_B{}^2-z_B{}^3+\sigma[z^2-(2z)z_B+z_B{}^2]+\frac{1}{27}[2\beta'^3-2(\beta'^2-3\gamma')\sigma-9\beta'\gamma'+27\delta']=+\beta' z^2+\gamma' z+\delta'$$

$$-(3z^2)z_B+(3z)z_B{}^2-z_B{}^3+\sigma[z^2-(2z)z_B+z_B{}^2]+\frac{1}{27}[2\beta'^3-2\sigma^3-9\beta'\gamma'+27\delta']=+\beta' z^2+\gamma' z+\delta'$$

Substituting for σ as follows:

$$z_B=\frac{1}{3}[-\beta'-\sqrt{\beta'^2-3\gamma'}]$$

$$=\frac{1}{3}[-\beta'+\sigma]$$

$$3z_B+\beta'=\sigma$$

$$-(3z^2)z_B+(3z)z_B{}^2-z_B{}^3+(3z_B+\beta')[z^2-(2z)z_B+z_B{}^2]+\frac{1}{27}[2\beta'^3-2(3z_B+\beta')^3-9\beta'\gamma']=+\beta' z^2+\gamma' z'$$

$$-(3z^2)z_B+(3z)z_B{}^2-z_B{}^3+(3z^2)z_B-(6z)z_B{}^2+3z_B{}^3+\beta' z^2-(2\beta' z)z_B+\beta' z_B{}^2+\frac{1}{27}[2\beta'^3-2(3z_B+\beta')^3-9\beta'\gamma']=+\beta' z^2+\gamma' z$$

$$-(3z)z_B{}^2+2z_B{}^3-(2\beta' z)z_B+\beta' z_B{}^2+\frac{1}{27}[2\beta'^3-2(3z_B+\beta')^3-9\beta'\gamma']-\gamma' z=0$$

$$-(3z)z_B{}^2+2z_B{}^3-(2\beta' z)z_B+\beta' z_B{}^2+\frac{1}{27}[2\beta'^3-2(27z_B{}^3+27\beta' z_B{}^2+9\beta'^2 z_B+\beta'^3)-9\beta'\gamma']-\gamma' z=0$$

$$-(3z)z_B{}^2+2z_B{}^3-(2\beta' z)z_B+\beta' z_B{}^2+\frac{1}{27}[-2(27z_B{}^3+27\beta' z_B{}^2+9\beta'^2 z_B)-9\beta'\gamma']-\gamma' z=0$$

$$-(3z)z_B{}^2+2z_B{}^3-(2\beta' z)z_B+\beta' z_B{}^2-2z_B{}^3-2\beta' z_B{}^2-\frac{2}{3}\beta'^2 z_B-\frac{1}{3}\beta'\gamma'-\gamma' z=0$$

$$-(3z)z_B{}^2-(2\beta' z)z_B-\beta' z_B{}^2-\frac{2}{3}\beta'^2 z_B-\frac{1}{3}\beta'\gamma'-\gamma' z=0$$

$$-(3z+\beta')z_B{}^2-(2\beta' z+\frac{2}{3}\beta'^2)z_B-\frac{1}{3}\beta'\gamma'-\gamma' z=0$$

$$-(3z+\beta')(\frac{1}{9})(\beta'^2+2\beta'\sqrt{\beta'^2-3\gamma'}+\beta'^2-3\gamma')-(2\beta' z+\frac{2}{3}\beta'^2)(\frac{1}{3})(-\beta'-\sqrt{\beta'^2-3\gamma'})-\frac{1}{3}\beta'\gamma'-\gamma' z=0$$

$$-(3z+\beta')(\frac{1}{9})(2\beta'^2+2\beta'\sqrt{\beta'^2-3\gamma'}-3\gamma')-(2\beta' z+\frac{2}{3}\beta'^2)(\frac{1}{3})(-\beta'-\sqrt{\beta'^2-3\gamma'})-\frac{1}{3}\beta'\gamma'-\gamma' z=0$$

$$(3z+\beta')(2\beta'^2+2\beta'\sqrt{\beta'^2-3\gamma'}-3\gamma')+(6\beta' z+2\beta'^2)(-\beta'-\sqrt{\beta'^2-3\gamma'})+3\beta'\gamma'+9\gamma' z=0$$

$$6\beta'^2 z+6\beta' z\sqrt{\beta'^2-3\gamma'}-9\gamma' z+2\beta'^3+2\beta'^2\sqrt{\beta'^2-3\gamma'}-3\beta'\gamma'+(6\beta' z+2\beta'^2)(-\beta'-\sqrt{\beta'^2-3\gamma'})+3\beta'\gamma'+9\gamma' z=0$$

$$6\beta'^2 z+6\beta' z\sqrt{\beta'^2-3\gamma'}+2\beta'^3+2\beta'^2\sqrt{\beta'^2-3\gamma'}+(6\beta' z+2\beta'^2)(-\beta'-\sqrt{\beta'^2-3\gamma'})=0$$

$$6\beta'^2 z+6\beta' z\sqrt{\beta'^2-3\gamma'}+2\beta'^3+2\beta'^2\sqrt{\beta'^2-3\gamma'}-6\beta'^2 z-6\beta' z\sqrt{\beta'^2-3\gamma'}-2\beta'^3-2\beta'^2\sqrt{\beta'^2-3\gamma'}=0$$

$$(6-6)\beta'^2 z+(6-6)\beta' z)\sqrt{\beta'^2-3\gamma'}+(2-2)\beta'^3+(2-2)\beta'^2\sqrt{\beta'^2-3\gamma'}=0$$

$$(0)\beta'^2 z+(0)\beta' z)\sqrt{\beta'^2-3\gamma'}+(0)\beta'^3+(0)\beta'^2\sqrt{\beta'^2-3\gamma'}=0$$

$$0=0$$

Hence, the above validation proves the two equations denoted below generate *identically shaped cubic functions*:

- $z^3+\beta' z^2+\gamma' z+\delta' = y_{TRANSFORMED}$
- $z'^3+\sigma z'^2+\nu = y'$

Naturally, the function $z'^3+\sigma z'^2+\nu = y'$ exhibits the same curve shape as $z'^3+\sigma z'^2 = y'-\nu = y''$ which rides either directly above or below it.

14.2.2. The Algorithm.

Generalized Cubic Curve Mapping is based upon the simple premise that a *Parent Generalized Cubic Function* exists which can fully characterize any *given Generalized Cubic Curve* in every respect.

Moreover, its *coefficient structure* can be determined by *mathematically interpreting* the *values* of certain ***properties*** exhibited by such *given Generalized Cubic Curve*, hereinafter to be denoted or referred to as *quintessential elements* itemized below *(Ref. Figure 19)*:

- z_P depicts the horizontal projection from any arbitrarily selected origin to a *single point of inflection* resident upon the *cubic curve*. This measurement is equal to −β/3 computed by taking the second derivative of the *cubic function* and setting it equal to zero as follows:

$$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$$

$$3z^2 dz + (2\beta' z)dz + \gamma' dz = dy$$

$$(6z_P)dz^2 + (2\beta')dz^2 = d^2 y$$

$$6z_P + 2\beta' = d^2 y / dz^2$$

$$= 0$$

$$6z_P = -2\beta'$$

$$z_P = -\frac{\beta'}{3}$$

- ***Single point of inflection*** denotes a *singular* point on the *cubic curve* where the actual *curve bend* changes from positive to negative, or vice versa. Only one *point of inflection* occurs because the second derivative shown above with respect to its variable 'z' is *linear* in form.
- ***Relative low points* and *high points*** on the *cubic curve* are represented by *two points of zero slope,* z_A *and* z_B *(Ref. Figure 19)*. Only two such points exist because the first derivative shown above with respect to its variable *'z'* is *quadratic* in form. Intrinsic to the very nature of the *Cubic Function*, its relative low point and high point both exhibit *zero slopes* that, in a sense, just so happen to lie parallel to one another. For *single term Cubic Functions* of the form $\alpha z^3 = y$, this represents a horizontal *point of symmetry* placed at the curve's actual low point, or high point, depending upon the sign of α.

For *binomial, three, and four term cubic functions*, such relative low and high points constitute actual low and high points when moving away from the point of inflection in opposite directions as follows:

A relative *high point* which resides to the *left* of the point of inflection is the actual high point left of that point of inflection;

A relative *low point* which resides to the *left* of the point of inflection is the actual low point left of that point of inflection;

A relative *high point* which resides to the *right* of the point of inflection is the actual high point right of that point of inflection;

A relative *low point* which resides to the *right* of the point of inflection is the actual low point right of that point of inflection

- The ***origin*** is any arbitrarily selected point within the grid, or envisioned coordinate system
- The ***z-axis*** represents a line going through the *origin* which is drawn parallel to the *two points of zero slope*. ***The z-axis distinguishes*** that the parameter 'z', not 'x' applies to *Cubic Function* variables presented in this treatise
- The ***y-axis*** represents a line which also passes through the *origin* drawn perpendicular to the *z-axis*, thereby establishing a mutually orthogonal coordinate system
- z_A represents the *horizontal projection* extending between the origin and the *point of zero slope* residing at Point A upon the *Cubic Curve*
- z_B represents the *horizontal projection* extending between the origin and the *point of zero slope* residing at Point B upon the *Cubic Curve*. In this case, z_B is portrayed as a negative quantity
- ***Δ*** depicts the *horizontal projection* between the *two points of zero slope* on the *Cubic Curve*. Its length is determined as follows:

$$
\begin{aligned}
\Delta &= z_A - z_B \\
&= \frac{1}{3}[-\beta' + \sqrt{\beta'^2 - 3\gamma'} - (-\beta' - \sqrt{\beta'^2 - 3\gamma'})] \\
&= \frac{1}{3}[-\beta' + \sqrt{\beta'^2 - 3\gamma'} + \beta' + \sqrt{\beta'^2 - 3\gamma'})] \\
&= \frac{2}{3}\sqrt{\beta'^2 - 3\gamma'}
\end{aligned}
$$

TRUE SCANS

- ***y_A*** represents the length of *vertical projection* extending between the origin and the *point of zero slope* residing at Point A upon the *Cubic Curve*. In this particular case, y_A is portrayed as a negative quantity
- ***y_B*** represents the length of *vertical projection* extending between the origin and the *point of zero slope* residing at Point B upon the *Cubic Curve*
- ***ε*** depicts the *vertical projection* between the *two points of zero slope* on the *cubic curve*. Its length is determined as follows:

$$z_A, z_B = \frac{9}{27}(-\beta' \pm \sqrt{\beta'^2 - 3\gamma'})$$

$$z_A{}^2, z_B{}^2 = [\frac{1}{3}(-\beta' \pm \sqrt{\beta'^2 - 3\gamma'})]^2$$

$$= \frac{1}{9}(\beta'^2 \mp 2\beta'\sqrt{\beta'^2 - 3\gamma'} + \beta'^2 - 3\gamma')$$

$$= \frac{3}{27}(2\beta'^2 \mp 2\beta'\sqrt{\beta'^2 - 3\gamma'} - 3\gamma')$$

$$z_A{}^3, z_B{}^3 = [\frac{1}{3}(-\beta' \pm \sqrt{\beta'^2 - 3\gamma'})]^3$$

$$= \frac{1}{27}[-\beta'^3 \pm 3\beta'^2\sqrt{\beta'^2 - 3\gamma'} - 3\beta'(\beta'^2 - 3\gamma) \pm (\beta'^2 - 3\gamma)\sqrt{\beta'^2 - 3\gamma'}]$$

$$= \frac{1}{27}[-4\beta'^3 \pm (4\beta'^2 - 3\gamma)\sqrt{\beta'^2 - 3\gamma'} + 9\beta'\gamma]$$

Then,

$$z_B{}^3 - z_A{}^3 = \frac{1}{27}[-(4\beta'^2 - 3\gamma)\sqrt{\beta'^2 - 3\gamma'} - (4\beta'^2 - 3\gamma)\sqrt{\beta'^2 - 3\gamma'}]$$

$$= \frac{1}{27}[-2(4\beta'^2 - 3\gamma')\sqrt{\beta'^2 - 3\gamma'}]$$

$$z_B{}^2 - z_A{}^2 = \frac{3}{27}(2\beta'\sqrt{\beta'^2 - 3\gamma'}) + 2\beta'\sqrt{\beta'^2 - 3\gamma'})$$

$$= \frac{3}{27}(4\beta'\sqrt{\beta'^2 - 3\gamma'})$$

$$\beta'(z_B{}^2 - z_A{}^2) = \frac{3}{27}(4\beta'^2\sqrt{\beta'^2 - 3\gamma'})$$

$$z_B - z_A = \frac{9}{27}(-\sqrt{\beta'^2 - 3\gamma'}) - \sqrt{\beta'^2 - 3\gamma'})$$

$$= \frac{9}{27}(-2\sqrt{\beta'^2 - 3\gamma'})$$

$$\gamma(z_B - z_A) = \frac{9}{27}(-2\gamma\sqrt{\beta'^2 - 3\gamma'})$$

TRUE SCANS

Accordingly,

$$\begin{aligned}
\varepsilon &= y_B - y_A \\
&= (z_B{}^3 + \beta' z_B{}^2 + \gamma' z_B + \delta') - (z_A{}^3 + \beta' z_A{}^2 + \gamma' z_A + \delta') \\
&= z_B{}^3 - z_A{}^3 + \beta'(z_B{}^2 - z_A{}^2) + \gamma'(z_B - z_A) + (\delta' - \delta') \\
&= z_B{}^3 - z_A{}^3 + \beta'(z_B{}^2 - z_A{}^2) + \gamma'(z_B - z_A) \\
&= \frac{1}{27}[(2)(3\gamma' - 4\beta'^2)\sqrt{\beta'^2 - 3\gamma'} + 12\beta'^2\sqrt{\beta'^2 - 3\gamma'} - 18\gamma'\sqrt{\beta'^2 - 3\gamma'}] \\
&= \frac{1}{27}(6\gamma' - 8\beta'^2 + 12\beta'^2 - 18\gamma')\sqrt{\beta'^2 - 3\gamma'} \\
&= \frac{1}{27}(4\beta'^2 - 12\gamma')\sqrt{\beta'^2 - 3\gamma'} \\
&= \frac{4}{27}(\beta'^2 - 3\gamma')\sqrt{\beta'^2 - 3\gamma'} \\
&= \frac{4}{27}(\beta'^2 - 3\gamma')^{\frac{3}{2}}
\end{aligned}$$

As indicated in *Figure 19*, Point P lies *midway* on the straight line which joins Point A to Point B *(validation given below)*. Now, the slope of that line is calculated to be:

$$\begin{aligned}
m &= -\frac{\varepsilon}{\Delta} \\
&= -\frac{\frac{4}{27}(\beta'^2 - 3\gamma')^{\frac{3}{2}}}{\frac{2}{3}\sqrt{\beta'^2 - 3\gamma'}} \\
&= -\frac{\frac{4}{27}(\sqrt{\beta'^2 - 3\gamma'})^3}{\frac{2}{3}\sqrt{\beta'^2 - 3\gamma'}} \\
&= -\frac{\frac{4}{27}(\sqrt{\beta'^2 - 3\gamma'})^2}{\frac{2}{3}} \\
&= -\frac{2}{9}(\beta'^2 - 3\gamma')
\end{aligned}$$

TRUE SCANS

Validation:

$$z_A, z_B = \frac{1}{3}[-\beta' \pm \sqrt{\beta'^2 - 3\gamma'}]$$

$$= z_P \pm \frac{1}{3}\sqrt{\beta'^2 - 3\gamma'}$$

Accordingly,

$$z_A - z_P = +\frac{1}{3}\sqrt{\beta'^2 - 3\gamma'} = z_P - z_B$$

Hence, the horizontal offset from Point A to Point P is exactly the same as that running from Point P to Point B.

Furthermore,

$$\begin{aligned}
y_P &= z_P{}^3 + \beta' z_P{}^2 + \gamma' z_P + \delta' \\
&= (-\frac{\beta'}{3})^3 + \beta'(-\frac{\beta'}{3})^2 + \gamma'(-\frac{\beta'}{3}) + \delta' \\
&= -\frac{\beta'^3}{27} + \frac{3\beta'^3}{27} - \gamma'(\frac{\beta'}{3}) + \delta' \\
&= \frac{1}{27}(2\beta'^3 - 9\gamma'\beta' + 27\delta')
\end{aligned}$$

$$\begin{aligned}
y_B &= z_B{}^3 + \beta' z_B{}^2 + \gamma' z_B + \delta' \\
&= \frac{1}{27}[-4\beta'^3 + (3\gamma' - 4\beta'^2)\sqrt{\beta'^2 - 3\gamma'} + 9\beta'\gamma'] + \frac{3\beta'}{27}[2\beta'^2 + 2\beta'\sqrt{\beta'^2 - 3\gamma'} - 3\gamma'] + \frac{9\gamma'}{27}(-\beta' - \sqrt{\beta'^2 - 3\gamma'}) + \frac{27}{27}\delta' \\
&= \frac{1}{27}[-4\beta'^3 + (3\gamma' - 4\beta'^2)\sqrt{\beta'^2 - 3\gamma'} + 9\beta'\gamma' + 3\beta'(2\beta'^2 + 2\beta'\sqrt{\beta'^2 - 3\gamma'} - 3\gamma') + 9\gamma'(-\beta' - \sqrt{\beta'^2 - 3\gamma'}) + 27\delta'] \\
&= \frac{1}{27}[-4\beta'^3 + (3\gamma' - 4\beta'^2)\sqrt{\beta'^2 - 3\gamma'} + 9\beta'\gamma' + 6\beta'^3 + 6\beta'^2\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' - 9\beta'\gamma' - 9\gamma'\sqrt{\beta'^2 - 3\gamma'} + 27\delta'] \\
&= \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta'] \\
&= \nu
\end{aligned}$$

$$\begin{aligned}
y_B - y_P &= \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\gamma'\beta' + 27\delta'] - \frac{1}{27}(2\beta'^3 - 9\gamma'\beta' + 27\delta') \\
&= \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\gamma'\beta' + 27\delta' - 2\beta'^3 + 9\gamma'\beta' - 27\delta'] \\
&= \frac{2}{27}(\beta'^2 - 3\gamma')\sqrt{\beta'^2 - 3\gamma'} \\
&= \frac{2}{27}(\beta'^2 - 3\gamma')^{\frac{3}{2}} \\
&= \frac{\varepsilon}{2}
\end{aligned}$$

TRUE SCANS

Now, considering that the *singularity proof* presented in *Section 14.2.1* validates that the two equations denoted below generate *identically shaped cubic functions*:

- $z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$
- $z'^3 + \sigma z'^2 + \nu = y'$

A third *parent equation* is to be introduced as follows:

- $z'^3 + \sigma z'^2 = y''$

It is easily recognized that $z'^3 + \sigma z'^2 = y''$ produces an identical curve to the *cubic function* $z'^3 + \sigma z'^2 + \nu = y'$ because for each and every value of z′ afforded, a value of y'' is produced, along with another respective value of y′ which is located a distance of ν above it. Accordingly, the y′ curve must trace out a curve which is *identical* to the y'' curve in every respect except that it maps out a locus which is displaced a distance of ν *vertically* above the y′ curve. . This positioning is represented in *Table 22*.

Table 22. Curve Plot of $z'^3+\sigma z'^2 = y''$ with respect to $z'^3+\sigma z'^2+\nu = y'$.

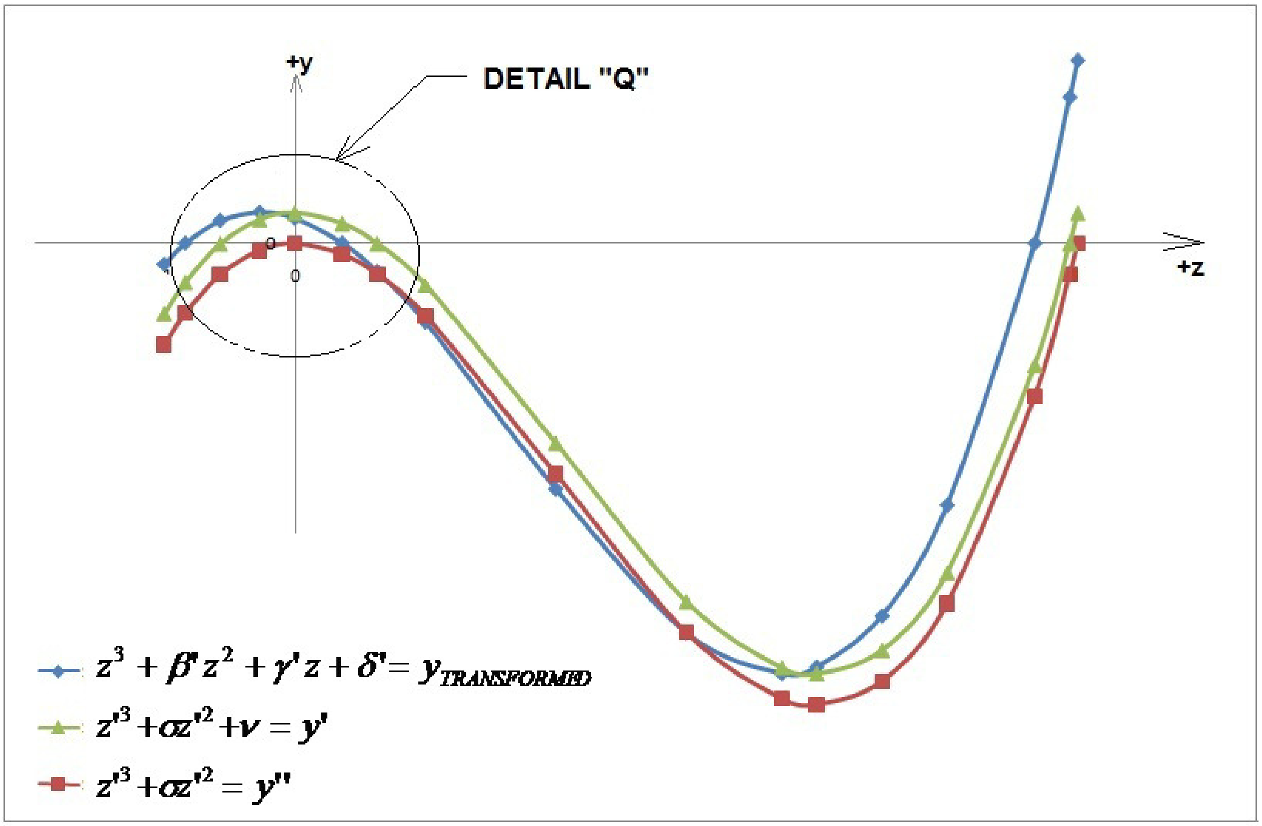
z = z′ Values	y'' **Values**	**y′ Values**
$z_1 = z_1'$	y''_1	$y''_1 + \nu$
$z_2 = z_2'$	y''_2	$y''_2 + \nu$
$z_3 = z_3'$	y''_3	$y''_3 + \nu$
$z_4 = z_4'$	y''_4	$y''_4 + \nu$
$z_5 = z_5'$	y''_5	$y''_5 + \nu$

Figure 20 illustrates the following three identically shaped curves which are displaced relative to one another:

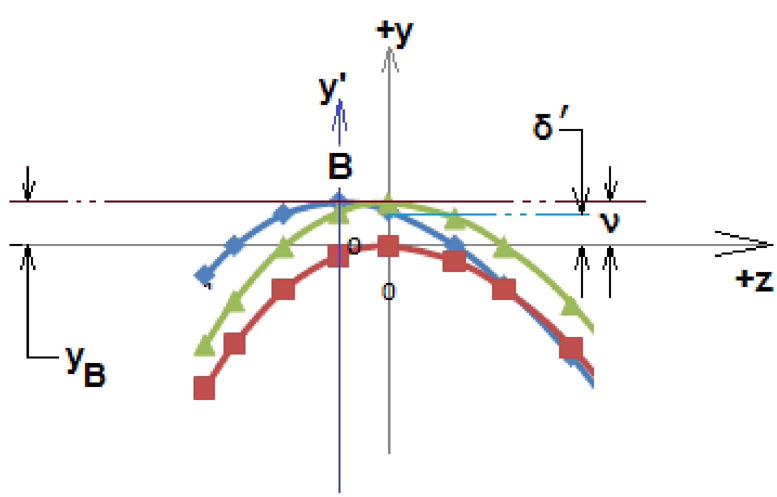
- $z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$
- $z'^3 + \sigma z'^2 + \nu = y'$
- $z'^3 + \sigma z'^2 = y''$

All curves exhibited in *Figure 20* are charted with respect to an origin which, for this particular case, has been arbitrarily selected at the juncture of the z-axis and the y-axis.

Figure 20. Set of Identical, but Displaced *Cubic Curves*.



DETAIL "Q"



The *Cubic Family Curve* has its relative high point of zero slope resting on the origin. This is because the first derivative of $z'^3 + \sigma z'^2 = y''$, when set to zero, locates it there.

The *Figure 20* detail shows that Point B, a location of *zero slope* for the $z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$ curve displays a y_B value which is exactly the same as the value of ν for the $z'^3 + \sigma z'^2 + \nu = y'$ curve. Again, as reflected in *Table 22*, this value of ν represents the distance which the $z'^3 + \sigma z'^2 + \nu = y'$ curve is elevated above its parent $z'^3 + \sigma z'^2 = y''$ curve.

When z equals zero, the $z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$ curve displays a vertical displacement of δ′ above the z-axis.

Lastly, the *Parent Cubic Function* $z'^3 + \sigma z'^2 = y''$ is easily relegated to an equation which may be determined from measurable *properties* intrinsic to the *given Cubic Curve* via taking its *second derivative* and setting it equal to zero as follows:

$$z'^3 + \sigma z'^2 = y''$$

$$(3z'^2)dz + (2\sigma z')dz = dy''$$

$$(6z'_P)dz^2 + (2\sigma)dz^2 = d^2y''$$

$$6z'_P + 2\sigma = d^2y''/dz^2$$

$$= 0$$

Or,

$$z'_P = -\frac{\sigma}{3}$$

$$= \frac{\Delta}{2}$$

So,

$$\sigma = -\frac{3\Delta}{2}$$

This determination applies because the *Parent Cubic Curve* has been shown to be *identical in shape* to the *given Cubic Curve*, and therefore possesses the same value for Δ/2. The *Parent Cubic Function* then reduces to $z'^3 - \frac{3\Delta}{2} z'^2 = y''$ and can be easily plotted to *exactly reproduce* the *given Cubic Curve*.

Now, the only question remaining is where to place the origin of the *given Cubic Curve* in order to represent its proper displacement. A threefold answer results as follows:

- If a drawing of the *given Cubic Curve* is accompanied by the relative location of its origin, then the newly drawn $z'^3 + \sigma z'^2 = y''$ *Parent Cubic Curve* should be situated with its *point B* placed at that same location
- If a drawing of the *given Cubic Curve* is not accompanied by the relative location of its origin, then such relative location doesn't figure into the assessment because a *new function* has been rendered which reproduces the exact same shape as the *given Cubic Curve*
- If a drawing of the *given Cubic Curve* is accompanied by a notation which describes its particular $z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$ *Function*, then the entire *Curve Mapping* exercise is unwarranted to begin with

14.2.3. Application.

From *properties* exhibited *in Cubic Curves*, an interpretation is now possible which permits intuitive associations and linkages.

Naturally, the first of such associations pertains to *Cubic Curve* relationships with respect to ζ. This is accomplished as follows for:

$$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$$

Taking the first derivative of the *function* for *Equation 23*, produces the following result:

$$z^3 - 3\zeta z^2 - 3z + \zeta = y_{TRANSFORMED} \qquad \textit{[Ref. Equation 22]}$$

$$(3z^2)dz - (6\zeta z)dz - 3dz = dy_{TRANSFORMED}$$

$$3z^2 - 6\zeta z - 3 = \frac{dy_{TRANSFORMED}}{dz}$$

Then,

$$z^2 - 2\zeta z - 1 = 0$$

$$z^2 - 2\zeta z + \zeta^2 = 1 + \zeta^2$$

$$(z - \zeta)^2 = 1 + \zeta^2$$

$$z_A, z_B = \zeta \pm \sqrt{1 + \zeta^2}$$

TRUE SCANS

$$\Delta = z_A - z_B$$
$$= \zeta + \sqrt{1+\zeta^2} - (\zeta - \sqrt{1+\zeta^2})$$
$$= 2\sqrt{1+\zeta^2}$$
$$z_P = \Delta / 2$$
$$= \sqrt{1+\zeta^2}$$
$$\sigma = -\frac{3\Delta}{2}$$
$$= -3\sqrt{1+\zeta^2}$$

Hence, a *Parent Cubic Curve* exists of the same exact shape as follows:

$$z'^3 + \sigma z'^2 = y''$$
$$z'^3 - 3\sqrt{1+\zeta^2}\, z'^2 = y''$$

Figure 21 displays various plots of this determined *Parent Cubic Function* for values of 3θ which range between 45° to 85°.

The *variability* in *Parent Cubic Function span*, as demonstrated by its change in Δ for related curves with differing 3θ values, is clearly evidenced in the fourth row of *Table 23*. The fifth row gives coefficient σ values for each of five respective 3θ-based *Cubic Curves*.

Figure 21. ***Cubic Parent Function*** **Association for ζ=tan 3θ.**

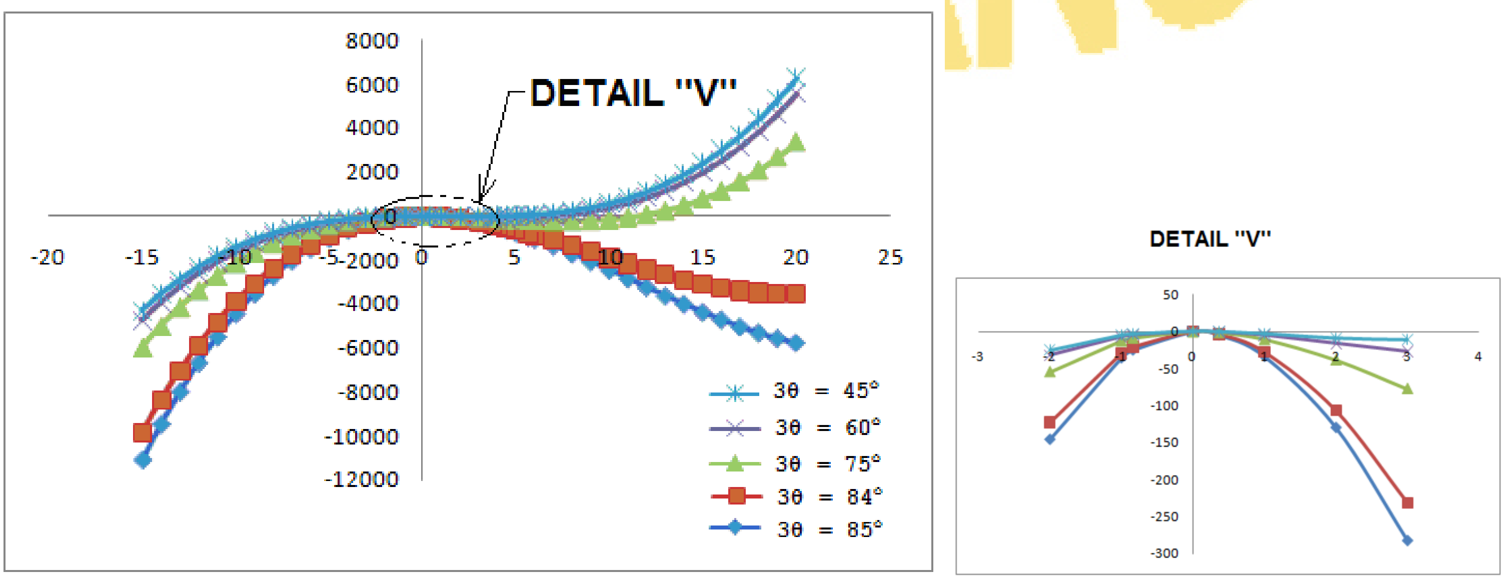


Table 23. Cubic Parent Function Association for ζ=tan 3θ Plot.

θ	28-1/3°	28°	25°	20°	15°
3θ	85°	84°	75°	60°	45°
ζ = tan 3θ	11.4300523	9.514364454	3.732050808	1.732050808	1
$z_P = \Delta/2 = \sqrt{1+\zeta^2}$	11.47371324	9.566772233	3.863703306	2	1.414213562
σ =-3 Δ/2	-34.42113973	-28.7003167	-11.59110992	-6.000000001	-4.242640687
z	$z'^3+\sigma z'^2=y''$	$z'^3+\sigma z'^2=y''$	$z'^3+\sigma z'^2=y''$	$z'^3+\sigma z'^2=y''$	$z'^3+\sigma z'^2=y''$
20	-5768.455892	-3480.12668	3363.556033	5600	6302.943725
19	-5567.031442	-3501.814329	2674.60932	4693	5327.406712
18	-5320.449272	-3466.902611	2076.480387	3888	4457.384417
17	-5034.709382	-3381.391526	1563.169234	3179	3686.876841
16	-4715.811771	-3251.281075	1128.675861	2560	3009.883984
15	-4369.756439	-3082.571257	767.0002687	2025	2420.405845
14	-4002.543387	-2881.262073	472.1424563	1568	1912.442425
13	-3620.172614	-2653.353522	238.1024241	1183	1479.993724
12	-3228.644121	-2404.845605	58.88017199	863.9999998	1117.059741
11	-2833.957907	-2141.738321	-71.52429992	604.9999999	817.6404769
10	-2442.113973	-1870.03167	-159.1109917	399.9999999	575.7359313
9	-2059.112318	-1595.725653	-209.8799033	242.9999999	385.3461043
8	-1690.952943	-1324.820269	-229.8310347	127.9999999	240.470996
7	-1343.635847	-1063.315518	-224.9643859	48.99999995	135.1106063
6	-1023.16103	-817.2114012	-201.279957	-4.03233E-08	63.26493526
5.67128182	-924.6942575	-740.6929236	-190.4020209	-10.57270667	45.95000976
5	-735.5284932	-592.5079175	-164.7777479	-25.00000003	18.93398282
4.5	-605.9030795	-490.0564132	-143.5949758	-30.37500002	5.211526086
4	-486.7382357	-395.2050672	-121.4577587	-32.00000002	-3.882250994
3.732050808	-427.4438675	-347.7630817	-109.4625725	-31.58845729	-7.111599605
3	-282.7902576	-231.3028503	-77.31998925	-27.00000001	-11.18376618
2	-129.6845589	-106.8012668	-38.36443967	-16	-8.970562748
1	-33.42113973	-27.7003167	-10.59110992	-5.000000001	-3.242640687
0.363970234	-4.511700753	-3.753838548	-1.487307821	-0.746629274	-0.513824274
0	0	0	0	0	0
-0.839099631	-24.82631814	-20.7983542	-8.751963751	-4.815329286	-3.577993346
-1	-35.42113973	-29.7003167	-12.59110992	-7.000000001	-5.242640687
-2	-145.6845589	-122.8012668	-54.36443967	-32	-24.97056275
-3	-336.7902576	-285.3028503	-131.3199893	-81.00000001	-65.18376618
-4	-614.7382357	-523.2050672	-249.4577587	-160	-131.882251
-5	-985.5284932	-842.5079175	-414.7777479	-275	-231.0660172
-6	-1455.16103	-1249.211401	-633.279957	-432	-368.7350647
-7	-2029.635847	-1749.315518	-910.9643859	-637.0000001	-550.8893937

θ	28-1/3°	28°	25°	20°	15°
3θ ζ = tan 3θ	85° 11.4300523	84° 9.514364454	75° 3.732050808	60° 1.732050808	45° 1
$z_P = \Delta/2 = \sqrt{1+\zeta^2}$	11.47371324	9.566772233	3.863703306	2	1.414213562
σ =-3 Δ/2	-34.42113973	-28.7003167	-11.59110992	-6.000000001	-4.242640687
z	$z'^3+\sigma z'^2=y''$	$z'^3+\sigma z'^2=y''$	$z'^3+\sigma z'^2=y''$	$z'^3+\sigma z'^2=y''$	$z'^3+\sigma z'^2=y''$
-8	-2714.952943	-2348.820269	-1253.831035	-896.0000001	-783.529004
-9	-3517.112318	-3053.725653	-1667.879903	-1215	-1072.653896
-10	-4442.113973	-3870.03167	-2159.110992	-1600	-1424.264069
-11	-5495.957907	-4803.738321	-2733.5243	-2057	-1844.359523
-12	-6684.644121	-5860.845605	-3397.119828	-2592	-2338.940259
-13	-8014.172614	-7047.353522	-4155.897576	-3211	-2914.006276
-14	-9490.543387	-8369.262073	-5015.857544	-3920	-3575.557575
-15	-11119.75644	-9832.571257	-5982.999731	-4725	-4329.594155

Conversely, the relative location of the *origin* for the given *Cubic Function* $z^3 - 3\zeta z^2 - 3z + \zeta = y_{TRANSFORMED}$ is determined as follows for the particular condition when:

- $\zeta = \tan(3\theta) = \sqrt{3}$
- $\Delta = 4 = 2\sqrt{1+\zeta^2}$

$$2 = \sqrt{1+\zeta^2}$$

$$z_A = \zeta + \sqrt{1+\zeta^2} = \zeta + 2 = 3.732050808$$

$$\begin{aligned} y_{A(TRANSFORMED)} &= z_A{}^3 - 3\zeta z_A{}^2 - 3z_A + \zeta \\ &= (\zeta+2)^3 - 3\zeta(\zeta+2)^2 - 3(\zeta+2) + \zeta \\ &= (7+4\zeta)(\zeta+2) - 3\zeta(7+4\zeta) - 3(\zeta+2) + \zeta \\ &= (7\zeta+14+12+8\zeta) - 21\zeta - 36 - 3\zeta - 6 + \zeta \\ &= -16 - 8\zeta \end{aligned}$$

$$\begin{aligned} \varepsilon &= \frac{4}{27}(\beta'^2 - 3\gamma')^{\frac{3}{2}} \\ &= \frac{4}{27}[(3\sqrt{3})^2 + 9]^{\frac{3}{2}} \\ &= \frac{4}{27}(36)^{\frac{3}{2}} \\ &= \frac{4}{27}(6)^3 \\ &= 32 \end{aligned}$$

TRUE SCANS

The location of the z-axis, relative to the given *Cubic Curve's* relative high and low points, is determined as a percentage $y_{A(TRANSFORMED)}/\varepsilon$ as follows:

$$\frac{y_{A(TRANSFORMED)}}{\varepsilon}=\frac{y_A}{\varepsilon}=\frac{-8(2+\zeta)}{8(4)}$$

$$=\frac{-(3.732050808)}{4}$$

$$=-0.933012701$$

Or,

$$y_A=-0.933012701\varepsilon$$

This indicates that y_A is equal to 93.3012701% the height of ε *(Ref. Figure 19)*.

Therefore, based upon the *triangular similarity* rendered below, the origin must lie on straight line APB *(Ref. Figure 19)*.

$$\frac{-z_A}{\Delta}=\frac{-3.732050808}{4}=-0.933012701=\frac{y_A}{\varepsilon}$$

This *ratio* afforded above applies to a *similarity* between *two right triangles* whose respective angles are equal, where:

- The larger triangle maintains a hypotenuse denoted by APB in *Figure 19*. It exhibits respective sides adjacent to its included right angle of Δ and ε, respectively
- The smaller similar triangle exhibits respective sides adjacent to its included right angle of $-z_A$ and y_A, respectively

For $\zeta=\sqrt{3}$, the *Parent Cubic Function* reduces to the following expression:

$$z'^3+\sigma z'^2=y''=z'^3-3\sqrt{1+\zeta^2}\,z'^2$$

$$=z'^3-3\sqrt{1+(\sqrt{3})^2}\,z'^2$$

$$=z'^3-3\sqrt{1+3}z'^2$$

$$=z'^3-3\sqrt{4}z'^2$$

$$=z'^3-3(2)z'^2$$

$$=z'^3-6z'^2$$

For $\zeta = \sqrt{3}$, the given *Family Cubic Function* reverts to the following:

$$z^3 - 3\zeta z^2 - 3z + \zeta = y_{TRANSFORMED}$$

$$z^3 - 3\sqrt{3}z^2 - 3z + \sqrt{3} = y_{TRANSFORMED}$$

$$z^3 - 3z + \sqrt{3}(1 - 3z^2) = y_{TRANSFORMED}$$ *[Ref. Equation 25]*

Equation 25 is plotted alongside its *Parent Curve*, noted above, in order to illustrate that they're identical.

For *Equation 25*, notice that $y_{TRANSFORMED} = \zeta = \sqrt{3} = 1.732050808$ when z equals zero, calculated as follows*:*

$$z^3 - 3z + \sqrt{3}(1 - 3z^2) = y_{TRANSFORMED}$$

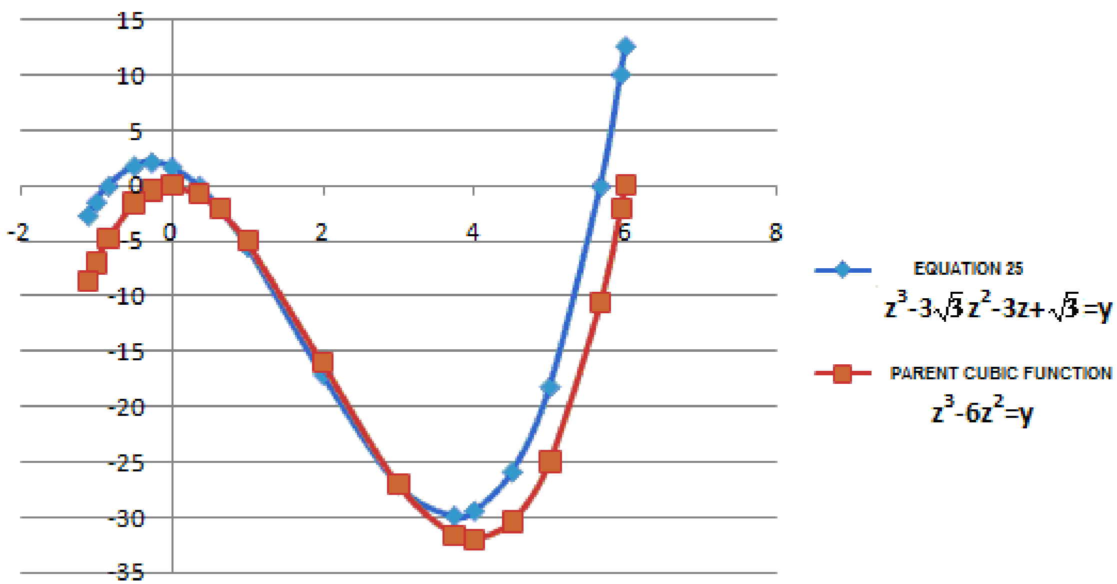
$$(0)^3 - 3(0) + \sqrt{3}[1 - 3(0)^2] =$$

$$\sqrt{3} =$$

Figure 22 shows the plot of *Equation 25* with respect to its *Parent Cubic Function*, thereby indicating identical curve shape.

TRUE SCANS

Figure 22. Equation 25 Plot and Assoc. *Parent Cubic Function* Identical Curve Shapes.



z	$z^3-3\sqrt{3}z^2-3z+\sqrt{3}=y$	$z^3-6z^2=y$
6	12.67056354	0
5.939231012	10.12605743	-2.143593541
5.67128182	0	-10.57270663
5	-18.17175979	-25
4.5	-25.86503578	-30.375
4	-29.40638798	-32
3.732050808	-29.85640648	-31.58845727
3	-27.03332101	-27
2	-17.05255889	-16
1	-5.464101616	-5
0.631919426	-1.986306854	-2.143593535
0.363970234	0	-0.746629274
0	1.732050808	0
-0.267949192	**2.14359354**	-0.450018503
-0.5	1.808012702	-1.625
-0.839099631	0	-4.815329286
-1	-1.464101616	-7
-1.107048823	-2.671735742	-8.71009412