THE PRINCIPLES OF

EQUATION SUB-ELEMENT THEORY

UNIQUE CAPABILITIES LIST

AND SUPPORTING DESCRIPTIONS

EQUATION SUB-ELEMENT THEORY
UNIQUE CAPABILITIES LIST

UNIQUE CAPABILITY DESCRIPTIONS

ITEM 1. Sub-elements Enable Diverse Equation Formats to become Categorized.

Diverse Quadratic and *Cubic equation formats* actually can become categorized with respect to one another.

Such formats become *associated* in terms of the following *factors* found to exist within *intrinsic root sets* and their *inherent coefficient structures*:

- $\tan\theta$
- $\tan(3\theta) = \zeta$

One specific example of an association between diverse *Quadratic* and *Cubic* Equation formats is provided below:

- The *Simplified Unified Cubic Trigonometric Reduction Equation (Ref. Equation 30)* represents a *Quadratic Equation* whose principal unknown is tan θ; where $\xi = \tan(3\theta)$ exists as a *factor* contained within both its first and third term *coefficients*:

  $$\zeta[C+3D]\tan^2\theta - [B-3D]\tan\theta - \zeta(D+1) = 0$$ *[Ref. Equation 30]*

- The *Generalized Cubic Equation (Ref. Equation 32)* exhibits *tan θ* as a *factor* inherent within *all three* of its *root values* z_R, z_S, and z_T *(see below);* wherein ζ manifests itself as an *Overall Equation Characteristic Value* that readily can be determined via manipulation of *Equation 32 coefficients* in accordance with *Equation 36*, also designated below:

$z^3 + \beta' z^2 + \gamma' z + \delta' = 0$	*[Ref. Equation 32, Sections 5.3 and 14.2]*

  Such that,

  $$\zeta = \frac{\delta' - \beta'}{1-\gamma'}$$ *[Ref. Equation 36]*

Accordingly, for each value of ζ identified in any given *Simplified Unified Cubic Trigonometric Reduction Equation*, there exists an *associated Generalized Cubic Equation* which features identical tan θ and $\xi = \tan(3\theta)$ properties.

Categorization becomes achieved because *Equation Sub-elements*, also deemed *RST terminology*:

- Appear as respective ***factors*** serving to *characterize Generalized Cubic Equation root set values* z_R, z_S, and z_T during specific circumstances when such equation's *coefficient* α is set equal to unity as follows:

$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$	*Generalized Cubic Equation [Ref. Equation 32]*

  $$z^3 + \beta z^2 + \gamma z + \delta = 0$$

  Where,

$z_R = R\tan\theta = \tan\theta_R$
$z_S = S\tan\theta = \tan\theta_S$ *[Ref. Section 10]*
$z_T = T\tan\theta = \tan\theta_T$

As indicated directly above, *RST terminology* furthermore relates the tangent of an *angle* θ to respective tangents of three *root set characteristic angles*, hereinafter denoted as θ_R, θ_S, *and* θ_T, the sum of which equals 3θ degrees as follows:

$$\theta_R + \theta_S + \theta_T = 3\theta$$ *[Ref. Section 10]*

Moreoever, such *Generalized Cubic Equation format*, as specified above, encompasses all *coefficient numerical variations* which could possibly be characterized by a *Cubic format* which consists of only a ***singular unknown***; one which furthermore assumes the form of a polynomial comprised of *cubic unknown* and *purely numeric terms*, in mathematical combination with either quadratic and/or linear *unknown* supporting terms.

Note:

The *Generalized Cubic Equation* format exhibits limited scope only from the standpoint that it addresses a ***singular cubic unknown***.

As such, it does not accommodate ***multiple sets*** of ***cubic unknowns*** such as z_1 and z_2, where each is afforded its own distinct *cubic root* values. An example of this type of ***complex cubic format*** is rendered below *(Ref. Section 2.2)*:

$$z_1^3 + \beta z_1^2 + \chi z_1 z_2^2 + \delta z_2^3 = 0$$

- Permeate, or are embedded deep within the framework, or architecture of constituent *algebraic equation coefficient structures*.

For example, the *Simplified Unified Cubic Trigonometric Reduction Equation format (Ref. Equation 30)* harbors *RST Terminology* within its inherent coefficients as follows:

$$\zeta[C+3D]\tan^2\theta - [B-3D]\tan\theta - \zeta(D+1) = 0$$ *[Ref. Equation 30]*

Such that,

$B = -(R+S+T)$
$C = RS + RT + ST$ *[Ref. Section 11.1]*
$D = -RST$

Such *categorization* is premised upon a *Hierarchy Chart* which eventually evolves into a *Characteristic Cubic Equation Thruway System* which are described as follows:

1.1. Hierarchy Chart.

Equation Sub-element *categorization begins* with the assemblage of a ***hierarchy chart*** *(Ref. Table 10)* that traces *paths of development* of certain *equations* and their *related functions*. It cites the exact source of each equation, and what specific type of *format* each represents consisting of the following classifications:

- Section 2. *Fundamental Information*
- Section 3. *Complex Quadratic Equations*
- Section 4. *Complex Quadratic Functions*
- Section 5. *Cubic Equations and Associated Functions*

Where,

Complex Quadratic Equations express combinations of first and second order *multiple unknown quantities* such as 'x_1', 'x_2', etc. *(Ref. Section 2.2)*. Such appellation is meant to distinguish them from regular, or so-called *normal Quadratic Equations* which *express* first and second order combinations of just a *singular* unknown quantity; in this case, 'x'.

Complex Quadratic Equations allow for *special monitoring* of *multiple unknowns* where each can become individually *interrogated*. This is similar to the manner in which *partial differential equations* may be used to *identify* specific values for typical *thermodynamic properties* such as pressure, volume, and density, by *acting upon one variable at a time* while ascribing distinct values to such other unknowns.

Such concept also extends itself to *Complex Linear Equations* which consist of more than a singular unknown, but express such *unknowns* only as linear quantities.

1.2. Characteristic Cubic Equation Thruway System.

Below is a list of supporting fundamental trans*forms which all stem from* a Unified Cubic Trigonometric Reduction Equation (*Ref. Equation 29 and Sections 11*):

- The *SUCTRE* -- see above *(Ref. Equation 30)*
- The *Characteristic Cubic Equation* *(Ref. Equation 31)*
- The *Generalized Cubic Equation* *(Ref. Equation 32)*
- The *Expression for S and T* *(Ref. Equation 33)*
- The *Expression for R and (S+T)* *(Ref. Equation 34)*
- The *Cubic Restitution Equation* *(Ref. Equation 35)*
- The *ζ Relationship to GCE Coefficients* *(Ref. Equation 36)*

Of these, the *Characteristic Cubic Equation (Ref. Equation 31) contains coefficients B, C, and D* which are ***inextricably linked*** to the other *transforms* via *RST Terminology*, reiterated as follows:

$$B = -(R + S + T)$$

$$C = RS + RT + ST$$

$$D = -RST$$

Such *B, C, and D coefficients* perform as *building blocks* that can be *associated* to a patchwork of other aggregate equation assemblages.

In a sense *Equation 31* may be viewed as a *crossroads* which interconnects a plethora of other *associated transforms* by means of a so-called ***Characteristic Cubic Equation Thruway System*** *(Ref. Section 12)*. It embodies various strategically emplaced *Quadratic and Cubic Equation Formats* where *travel* between respective points occurs whenever one format becomes successfully *transformed* into an adjoining one *(Ref. Table 16)*. The process is controlled by a *rigid set of rules (Ref. Table 17)*, each of which is comprised of a sequence of calculations which need to be determined in order to remain in compliance.

Such ***Thruway System*** may be compared favorably to the *software and codes* which led to the development of **relational databases**, now relied upon heavily in the field of *computer science*. For purposes of introducing *spreadsheets*, such *relational databases* first assumed the form of *System R* in its infancy; but later evolved into SQL, Oracle, and *Excel.*

ITEM 2. Sub-element Theory Proposes a Rationally-based Number Classification.

<u>All</u> *real numbers* can be categorized either as *rationally-based* or *cubic irrational*, where (*Ref. Section 9.1)*:

Rationally-based numbers consist of:

a. *All rational numbers; and*

b. *Quadratic irrational numbers* such as $17\sqrt{35\sqrt{7}/1025}$ which are comprised of the magnitudes of all lengths which can be geometrically constructed from a given length of unity other than those which are of rational value. When algebraically expressed, they must exhibit at least one square root radical sign. However, *quadratic irrational numbers* cannot feature any radical sign which is a multiple of three, such as a cube root or even possibly an eighty-first root, because such values cannot be determined by means of applying successive Quadratic Formulas that are permitted to operate only upon either rational numbers and/or quadratic equation root values, as might become determined by such method.

Cubic irrational numbers consist of <u>all</u> other real numbers that *cannot* be classified as *rationally-based*.

The *rationally-based number* classification should be viewed as a set of *real numbers* which includes all possible *Euclidean* determinations that can be ***geometrically constructed*** from a given, arbitrary length of unity.

It *collates* a disparate assortment of *rational* and *quadratic irrational lengths* together, like $4+(32/62)\sqrt{5}+17\sqrt{35\sqrt{7}/1025}$, whose *individual terms* consist specifically of:

1) *Rational numbers* -- defined as the *quotient* between two *given* integers, and portrayed as follows:

$$x_1 = \frac{\Delta}{2a} = \frac{x_1}{1}$$

Where the *mathematic division* represented above identifies a length x_1 that is determined via ***geometric construction*** performed in accordance with the *Euclidean Mapping Process* specified in *Section 2.3* whose:

- Lengths *Δ* and *2a*, each representing *integer values*, are ***geometrically constructed*** via sole *straightedge* and *compass* using an arbitrary, assigned length of *unity* as a basis
- *Rational length* x_1 is identified as the horizontal offset measured from the right side of the rectangle to the point where the diagonal line intersects the horizontal line which exhibits a height of unity *(Ref. Figure 2)*

Hence, all *rational numbers* are *Euclidean*! In other words, each and every one can be ***geometrically constructed*** from an arbitrary length which is to be designated as one unit in length via only a *straightedge* and *compass*; and

2) *Quadratic irrational numbers* – represented as magnitudes of *all* lengths that can be ***geometrically constructed*** via *Pythagorean Theorem* either from *solely rational lengths* in concert with an infinite variety of mathematical combinations of other *purely rational lengths*, or from their results.

Even after such *rational values* become *transformed* into *irrational lengths* via *Pythagorean Theorem*, it still remains possible to measure them, as well as to replicate them from a given, arbitrary length of unity.

Mathematically, such ***geometric construction*** process is analogous to calculating respective *root pair values* x_1 *and* x_2 depicted below via *Quadratic Formula* that operates *only* upon sole *rational (or rationally-based) coefficient values* a, b, and c that are inherent to, or reside within the specific *Quadratic Equation format* $ax^2+bx+c=0$:

$$x_1;x_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

In conclusion:

- ***Rationally-based numbers*** comprise all *real numbers* which can be ***geometrically constructed*** from a given, arbitrary length of unity
- ***Cubic irrational numbers*** comprise all other *real numbers*; specifically, those which cannot be ***geometrically constructed*** from a given, arbitrary length of unity

ITEM 3. Sub-element Theory Elucidates upon a Cubic Irrational Gateway

Earlier, a case is made that *rationally-based numbers* cannot beget *cubic irrational* ones.

In actuality, however, the reverse it true; whereby *cubic irrational root sets* can produce *rationally-based* results!

What's ***profound*** is that such *gateway* exists, not through *quadratic roots* and their associated *equations*, but only through *cubic root sets* where, yet again, the *Generalized Cubic Equation* takes center stage for the same reasons as mentioned earlier.

Such assertion, once proven true, gives *equation formats* ***meaning***; thereby, breathing new life into them.

Specifically, the *Generalized Cubic Equation* format enables *cubic irrational root set quantities* to co-exist within a *coefficient framework* comprised solely of *rationally-based numbers*.

This occurs through a ***mathematical gateway*** which becomes enabled by either calculating the *product*, *summation*, or *summation of paired products* of such aforementioned *cubic irrational triads (Ref. Tables 13 through 15)*.

Such gateway does not apply to *Quadratic Equations* of the reconstituted form $x^2+b'x+c'=0$, simply because they neither have the affinity, nor possess the capability to convert *cubic irrational* numbers into *rationally-based values*.

This is demonstrated via the *Quadratic Formula* as shown below; wherein respective *roots* x_1 *and* x_2 clearly cannot be *cubic irrational* when *coefficients b' and c'* are *rationally-based*:

$$x_1;x_2=\frac{-b'\pm\sqrt{b'^2-4c'}}{2}$$

The above analysis considers only *Quadratic and Cubic Equations* which express *singular* unknown variables. Hence, it does not address *Complex Quadratic and Cubic Equations*.

3.1. *The Cubic Equation Uniqueness Theorem.*

The above account gives rise to the ***Cubic Equation Uniqueness Theorem*** *(Ref. Section 9.3)*. As stated below, it applies exclusively to *equation formats* of ***singular unknown*** quantity such as:

- The *Generalized Cubic Equation Format* $z^3+\beta''z^2+\gamma''z+\delta''=0$
- The *Quadratic Equation Format* $x^2+b'x+c'=0$

> ***Only Cubic Equations allow solely rationally-based numerical coefficients to co-exist with root sets comprised of cubic irrational numbers.***

In other words, a unique capability to characterize *cubic irrational roots* in terms of solely *rationally-based coefficients must* be reserved only for *Cubic Equation formats.*

Accordingly, *cubic irrational roots* apply to *Cubic Equations* which contain *solely rationally-based coefficients*, but not to *Quadratic Equations* whose *coefficients* also are solely *rationally-based.*

Based upon this hypothesis, next consider the seemingly outlandish possibility that various *equation formats* might actually assume their very own form, or acquire their overall *algebraic aspect*, in order *to* account for the various types of *rationally-based* and *cubic irrational number arrangements* inherent within their very *coefficient* and *root structures*.

To reiterate, such ***formats*** harbor distinctively *different* arrangements or combinations of such *numerical representations* that each contains.

In this regard, ***Equation Sub-element Theory*** *distinguishes* between *Quadratic and Cubic Equation formats* and explains why *diversity* exists between them *(Ref. Section 9).*

It contends that *Cubic Equation formats* pose a complete *demarcation* from their *Linear* and *Quadratic Equation counterparts*. This is because they *must exist* as separate *mathematical entities, independent or completely apart* from *Quadratic Equation formats*, in order *allow* for a unique correlation between *rationally-based coefficients* and their associated *cubic irrational root sets.*

3.2. *A Corollary to the Cubic Equation Uniqueness Theorem.*

A principal ***corollary*** to the ***Cubic Equation Uniqueness Theorem*** specifically states *(Ref. Section 9.3)*:

> ***Cubic irrational root pairs* which appear in *Parabolic Equations* or their *associated functions* require *supporting cubic irrational coefficients*.**

3.3. *Rationale for a Cubic Irrational Mathematics Law.*

The complete *demarcation* between *rationally-based numbers* and *cubic irrational numbers* can be perceived ***mathematically***.

It appears, or becomes evident in various equation forms which very surprisingly are considered to be *mathematically correct*!

This occurs when attempts are made to *mathematically equate cubic irrational* results on one side of an equation in terms of sole *rationally-based* values enlisted upon the other!

When mathematical resolution cannot possibly be achieved, such equations instead manifest themselves as ***null sets***; i.e. ones which *appear* to condone an infinite supply of values as being *mathematically correct (Ref. Section 8)*.

In reality, though, some values may turn out to be ***incorrect***, but such equations really don't understand this because constraints imposed by other equations haven't yet been applied. Hence, such *null set results* really provide indication of an *insufficient amount of information*.

One such example is shown below for the particular circumstance when it is desired to establish a second, independent *Generalized Cubic Equation* which can be used in conjunction with a given *Generalized Cubic Equation*, by virtue of its *common root* z_R, in order to simultaneously resolve it.

The first given *Generalized Cubic Equation* is of the following form:

$z^3+\beta' z^2+\gamma' z+\delta'=0$	*[Ref. Equation 32, and Section14.2]*

Given $3\theta=60^o$, $R=2$, and $S=3$:

Values for ζ, tan θ and the associated roots are computed as follows:

$$\zeta=\tan(3\theta)=\tan 60^o=\sqrt{3}$$

$$\tan\theta=\tan(\frac{3\theta}{3})=\tan\frac{60^o}{3}=\tan 20^o=0.363970234$$

$$z_R=R\tan\theta=2\tan 20^o=0.727940468=\tan\theta_R=\tan 36.05238873^o$$
$$z_S=S\tan\theta=3\tan 20^o=1.091910703=\tan\theta_S=\tan 47.51574349^o$$

Such that:

$$3\theta=60^o=\theta_R+\theta_S+\theta_T \qquad \textit{[Ref. Item 1 Capabilities]}$$

$$=36.05238873^o+47.51574349^o+\theta_T$$

$$60^o-(36.05238873^o+47.51574349^o)=\theta_T$$

$$-23.56813222^o=\theta_T$$

$$\tan(-23.56813222^o)=\tan\theta_T=z_T=0.436227058=T\tan\theta=-1.198523992\tan 20^o$$

$$\begin{aligned}\beta' &= -(z_R + z_S + z_T) \\ &= -(2 + 3 - 1.198523992)\tan 20^o \\ &= -3.801476008(0.363970234) \\ &= -1.383624113\end{aligned}$$

$$\begin{aligned}\gamma' &= z_R(z_S + z_T) + z_S z_T \\ &= 2\tan 20^o(3 - 1.198523992)\tan 20^o - 3\tan 20^o(1.198523992\tan 20^o) \\ &= [2(3 - 1.198523992) - 3(1.198523992)]\tan^2 20^o \\ &= 0.00738004(0.363970234)^2 \\ &= 0.000977665\end{aligned}$$

$$\begin{aligned}\delta' &= -z_R z_S z_T \\ &= 2\tan 20^o(3\tan 20^o)(1.198523992\tan 20^o) \\ &= 2(3)(1.198523992)\tan^3 20^o \\ &= 7.191143952(0.363970234)^3 \\ &= 0.346733327\end{aligned}$$

Check,

$$\zeta = \frac{\delta' - \beta'}{1 - \gamma'} \qquad \text{[Ref. Equation 36]}$$

$$\begin{aligned}&= \frac{0.346733327 + 1.383624113}{1 - 0.000977665} \\ &= 1.732050808 \\ &= \sqrt{3}\end{aligned}$$

Such second, independent *Generalized Cubic Equation* is to be structured as follows:

$$\begin{aligned}3\theta &= \theta_R + \theta_{S'} + \theta_{T'} \\ 60^o &= \theta_R + \theta_{S'} + 15^o \\ 45^o &= \theta_R + \theta_{S'} \\ \tan 45^o &= \tan(\theta_R + \theta_S) \\ 1 &= \frac{z_R + z_{S'}}{1 - z_R z_{S'}} \\ 1 - z_R z_{S'} &= z_R + z_{S'} \\ 1 - z_R &= z_{S'}(1 + z_R) \\ \frac{1 - z_R}{1 + z_R} &= z_{S'}\end{aligned}$$

$z^3+\beta''z^2+\gamma'z+\delta''=0$	*[Ref. Equation 32, and Section 14.2]*

Where,

$$\beta''=-(z_R+z_{S'}+z_{T'})$$

$$=-[z_R+\frac{1-z_R}{(1+z_R)}+\tan 15^\circ]$$

$$=-\frac{1}{1+z_R}[z_R+z_R^2+1-z_R+(1+z_R)\tan 15^\circ]$$

$$=-\frac{1}{1+z_R}[z_R^2+z_R\tan 15^\circ+(1+\tan 15^\circ)]$$

$$\beta''(1+z_R)=-[z_R^2+z_R\tan 15^\circ+(1+\tan 15^\circ)]$$

$$\gamma''=z_R(z_{S'}+z_{T'})+z_{S'}z_{T'}$$

$$=z_R(\frac{1-z_R}{1+z_R}+\tan 15^\circ)+\frac{1-z_R}{1+z_R}\tan 15^\circ$$

$$=\frac{1}{1+z_R}[z_R-z_R^2+z_R(1+z_R)\tan 15^\circ+(1-z_R)\tan 15^\circ]$$

$$=\frac{1}{1+z_R}[z_R^2(\tan 15^\circ-1)+z_R+\tan 15^\circ]$$

$$\gamma''(1+z_R)=z_R^2(\tan 15^\circ-1)+z_R+\tan 15^\circ$$

$$\delta''=-z_Rz_{S'}z_{T'}$$

$$=-z_R\frac{1-z_R}{1+z_R}\tan 15^\circ$$

$$=-\frac{1}{1+z_R}(z_R-z_R^2)\tan 15^\circ$$

$$-\delta''(1+z_R)=(z_R-z_R^2)\tan 15^\circ$$

Such that,

$$\zeta=\frac{\delta''-\beta''}{1-\gamma''} \qquad \textit{[Ref. Equation 36]}$$

$$\zeta(1-\gamma'')=\delta''-\beta''$$

$$\zeta+\beta''-\delta'=\zeta\gamma''$$

$$(1+z_R)(\zeta+\beta''-\delta')=\zeta\gamma''(1+z_R)$$

$$\zeta(1+z_R)-[z_R^2+z_R\tan 15^\circ+(1+\tan 15^\circ)]+(z_R-z_R^2)\tan 15^\circ=\zeta[z_R^2(\tan 15^\circ-1)+z_R+\tan 15^\circ]$$

$$\zeta-[z_R^2+z_R\tan 15^\circ+(1+\tan 15^\circ)]+(z_R-z_R^2)\tan 15^\circ=\zeta[z_R^2(\tan 15^\circ-1)+\tan 15^\circ]$$

$$\zeta-z_R^2-z_R\tan 15^\circ-(1+\tan 15^\circ)+(z_R-z_R^2)\tan 15^\circ=\zeta[z_R^2(\tan 15^\circ-1)+\tan 15^\circ]$$

$$\zeta-z_R^2-(1+\tan 15^\circ)-z_R^2\tan 15^\circ=\zeta[z_R^2(\tan 15^\circ-1)+\tan 15^\circ]$$

$$\zeta-(1+\tan 15^\circ)-z_R^2(1+\tan 15^\circ)=\zeta z_R^2(\tan 15^\circ-1)+\zeta\tan 15^\circ$$

$$\zeta-(1+\tan 15^\circ)-\zeta\tan 15^\circ=z_R^2(\zeta\tan 15^\circ-\zeta+1+\tan 15^\circ)$$

$$\sqrt{\frac{\zeta-(1+\tan 15^\circ)-\zeta\tan 15^\circ}{\zeta\tan 15^\circ-\zeta+1+\tan 15^\circ}}=z_R$$

In addition,

$$\tan 15^{\circ} = \tan(60^{o} - 45^{o}) = \frac{\zeta - 1}{1 + \zeta}$$
$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

As indicated above, $z_R = R\tan\theta = 2\tan\theta = 2\tan 20^{\circ} = 2(0.363970234)$, being a quantity which is two times that of *tan 20*^{o}, is most definitely a *cubic irrational number*.

However, very surprisingly, it is *mathematically equated* to the *square root* of a certain combination of completely *rationally-based numbers* which, in turn, must constitute an overall ***rationally-based* result**!

Being that the above math is considered to be correct, this is clearly **impossible** simply because a *cubic irrational number* cannot be set equal to a ***rationally-based* result**!

This is *explained* by completing such equivalency as follows:

Where,

$$z_R = \sqrt{\frac{\zeta - 1 - \tan 15^{\circ}(1+\zeta)}{\tan 15^{\circ}(1+\zeta) - \zeta + 1}}$$
$$= \sqrt{\frac{(\zeta - 1) - (\frac{\zeta - 1}{1+\zeta})(1+\zeta)}{(\frac{\zeta - 1}{1+\zeta})(1+\zeta) - \zeta + 1}}$$
$$= \sqrt{\frac{(\zeta - 1) - (\zeta - 1)}{(\zeta - 1) - (\zeta - 1)}}$$
$$= \sqrt{\frac{0}{0}}$$
$$= NullSet$$

This *dramatic* result signifies the *nomenclature* that is elicited when such type of aforementioned *mathematics* impossibility is considered to eventuate.

Another way of viewing this ramification is as follows:

From above,

$$\zeta - (1 + \tan 15^{\circ}) - \zeta\tan 15^{\circ} = z_R^{\,2}(\zeta\tan 15^{\circ} - \zeta + 1 + \tan 15^{\circ})$$
$$(\zeta - 1) - \tan 15^{\circ}(1+\zeta) = z_R^{\,2}[\tan 15^{\circ}(1+\zeta) - (\zeta - 1)]$$
$$(\zeta - 1) - (\frac{\zeta - 1}{1+\zeta})(1+\zeta) = z_R^{\,2}[(\frac{\zeta - 1}{1+\zeta})(1+\zeta) - (\zeta - 1)]$$
$$(\zeta - 1) - (\zeta - 1) = z_R^{\,2}[(\zeta - 1) - (\zeta - 1)]$$
$$0 = 0 z_R^{\,2}$$

[Ref. Section 8.4]

Such result is considered to mean that any value which the *cubic root* z_R assumes still satisfies the equality!

Again, this is an indication that an inadequate amount of detail has been supplied.

Both above analyses indicate that such mathematically based, perpetuated *null set* nomenclature is ***completely influenced*** by a *far more important* governing rule which postulates that *cubic irrational numbers cannot* be represented via ***geometric construction*** predicated upon a given, arbitrary length of unity! In other words, they cannot *be geometrically constructed from* any *rationally-based* ***result***!

In addition to the *null set phenomenon* described above, such proposed ***governing rule*** also has been shown to exert a ***profound influence*** upon discerning a *cubic irrational gateway*.

Such accrued ***influence*** upon the overall field of mathematics is considered to be ***far more reaching*** than even that imposed by the very ***Law of Sines*** or ***Law of Cosines***!

Hence, it is proposed that such ***governing rule*** most probably now should be enacted into Law; thereby stipulating the *Section 9.1* reconstituted *conclusion* as follows:

- ***Rationally-based numbers*** comprise all *real numbers* which can be ***geometrically constructed*** from a given, arbitrary length of unity.
- ***Cubic irrational numbers*** comprise all other *real numbers*; specifically, those which cannot be ***geometrically constructed*** from a given, arbitrary length of unity

Where,

Geometrically construct denotes a capability to ascribe length by means of the *Euclidean Mapping Process* stipulated in *Section 2.3* of this treatise.

ITEM 4. Sub-element Theory Portrays Cubic Irrational Lengths from Unity and Trisects.

Equation Sub-element Theory features ***Atacins***, a novel invention which portrays *(Ref. Sections 22* and *22.6)*:

- ***Geometrically formed*** *cubic irrational lengths* from any arbitrarily assigned or given length of unity, while still adhering to all of the precepts espoused in the *conclusion* given in *Section 9.1*.
- ***Geometrically formed*** angles of *exactly one-third* the respective magnitudes of any *given angles*

Atacins is an *acronym* for ***a****ngle* ***t****risector* ***a****nd* ***c****ubic irrational length* ***i****nstrument* whereby a motion needs to be *imparted* in order to use it properly.

Such new device is to perform the ***principal function*** of identifying *cubic irrational lengths* first and foremost, while secondly performing the *lesser function* of actually *trisecting* various ascribed angles of size 3θ.

This priority is urged because the concept of depicting *exact cubic irrational lengths* alongside an amalgamation of *rationally-based lengths* that actually define them should exemplify a fitting or fundamentally new ***Number Theory*** groundwork; one from which amazing, new discovery may be launched, and one which should serve to appreciably *advance* the overall state-of-the-art!

In contrast, an ability to *trisect an angle*, although of *significant import*, nevertheless does not exemplify this same *profound capability* to stand alone as an actual ***Number Theory*** groundwork in itself; one from which other meaningful applications could then become derived.

4.1. *Geometrically Formed Cubic Irrational Lengths.*

Geometrically formed cubic irrational lengths become evident during ***overlapment***, a singular condition observed to occur whenever the *longitudinal axis* of a pre-selected *Atacins* compass arm hovers *directly over* the *determinable point (η,τ) (Ref. Sections 22.1)*.

How such condition occurs is thoroughly explained by an accompanying proof *(Ref. Sections 22.6.2)*.

Cubic irrational lengths result because *geometric constraint* becomes imposed upon the *endpoint* of the other compass arm.

Setting all *Atacins compass arm* and *straightedge* lengths equal to an arbitrary value of *unity* assures that resulting *cubic irrational lengths* can become portrayed directly alongside such *rational unitary basis*.

4.2. Theory.

Atacins physically discerns *cubic irrational lengths* indicative of *mathematical cubic root values* z_R, z_S, and z_T inherent within *3θ Cubic Equations* whose $\zeta = \tan(3\theta)$ values specifically consist of *(Ref. Section 22.5)*:

a) *Rationally-based lengths (Ref. Section 9.1)*; or
b) *Cubic irrational root lengths* ascertained from them.

In consonance with the *Cubic Equation Uniqueness Theorem*, reiterated below, this may be interpreted to mean *(Ref. Section 9.3)*:

> "*Only Cubic Equations allow solely rationally-based numerical coefficients to co-exist with root sets comprised of trigonometric, cubic irrational numbers".*
>
> When a ***3θ Cubic Equation***, of the particular form designated below, possesses a *rationally-based coefficient* of $\zeta = \tan(3\theta)$, its *roots* nevertheless still may be *cubic irrational*:
>
> $$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

During such circumstances, a ***co-existence*** between equation *rationally-based coefficients* and *associated cubic irrational roots* presumably occurs.

Table 35 relates how *cubic irrational root length values* ascertained from such specific *rationally-based values* become commissioned as actual ζ values in themselves, in order to perpetuate *numerical length determinations*.

4.3. Operation.

Atacins features only *compass* and *straightedge* construction where actuation proceeds from completely ***identifiable locations*** *(Ref. Section 22.7)*.

It enables the *trisector* of *any given angle* to be ***geometrically formed***, as opposed to *geometrically constructed*, simply by applying the following two step process *(Ref. Figure 51)*:

1) Set *angles AOB and A'O'B'* to *predetermined* angles of *90-3θ degrees* each*;*
2) Then articulate, or flex the invention until such time that the *longitudinal axis* of *member* $\overline{O'B'}$ **overlaps** *point B*.

The trisected *angle OO'C* thereafter becomes easily identified by *bisecting* the ***geometrically formed*** *angle OO'A'* either by use of added pencil/paper or via ruler ***(In the event of any conflict between this section and U.S. Patent No. 10994569 issued on 5/4/2021, the latter shall govern)***.

Atacins creates a ***geometrically formed*** depiction of an angle exactly one-third the magnitude of any *given angle* that is programmed into it. Even when the tangent of such *resulting angle* is a *cubic irrational length*, **Atacins** depicts it *(Ref. Section 22.6)*.

The device *overcomes* the *rational number* to *cubic irrational number quandary* normally experienced during prior attempts to perform *Euclidean trisection*.

This is achieved by articulating such invention until ***overlapment***, as described above, occurs; whereby, *cubic irrational lengths* become ***portrayed*** alongside *given rationally-based* ones.

During such articulation, compass endpoint A′ is to be constrained within the *slot arrangement* appearing in compass arm $\overline{OA}$, thereby permitting it to *ride* only in the horizontal direction, or *actuate* only along the x-axis *(Ref. Figure 51)*.

Atacins features *straightedge member* $\overline{OO'}$ whose *endpoints* interconnect to two hinges which belong to compasses OAB and O′A′B′, respectively *(Ref. Figure 51)*.

Therein, *members* $\overline{AB}$ and $\overline{A'B'}$ *extended* have been inserted only to replace the *tightening capabilities* of such respective compass hinges. Such modification simplifies the operation of the device, but is not mandatory.

Accordingly, *Atacins* consists of *two hinges which* attach the endpoints of a *middle straightedge* to respective *assemblies* of *swinging arms* which collectively may be actuated as *independent compasses.*

4.4. Reconciliation between Geometrically Formed Depictions and Proposed Law.

The above findings remain consistent with the *conclusion* expressed in *Section 9.1*, thereafter recommended to become enacted into *Law (Ref. Item 3.3)*, because of the fact that:

- ***Rationally-based numbers*** comprise all *real numbers* which can be ***geometrically constructed*** from a given, arbitrary length of unity
- ***Cubic irrational numbers*** comprise all other *real numbers*; specifically, those which cannot be ***geometrically constructed*** from a given, arbitrary length of unity, but can be ***geometrically formed*** from a given, arbitrary length of unity *(Ref. Section 22.7)*.

ITEM 5. Sub-element Theory Explains why Cubic Functions Indicate Relative Position.

The *Quadratic Formula* expresses *root set pairs*, designated below as x_1 and x_2, as little more than ***mathematical manipulations*** of only *intrinsic coefficients a, b,* and *c* ***harbored*** within *Parabolic Equations* of the form $ax^2+bx+c=0$:

$$x_1;x_2=[-b\pm\sqrt{b^2-4ac}]/2a$$

The ***Characteristic Cubic Equation Thruway*** *System* enhances upon this *practice* by enabling *mathematical operations* to be performed upon associated *equation formats* through a *conversion process*, or *transformation* which internally ***links*** resident *coefficient structures (Ref. Table 16)*.

Curve Mapping instead *mathematically operates* upon just one particular *coefficient* structure, or equation *format* at a time *(Ref. Section 14)*. It determines *sets*, or *families* of *coefficient permutations* comprised of intrinsic ***RST terminology***. Hence, a gateway for ***Equation Sub-element*** *categorizations* becomes realized.

Equation Sub-element Curve Mapping Theory maintains that a *stationary parabolic or Generalized Cubic curve shape* exhibits a singular ***equation format structure*** but, nevertheless, may be characterized by a *multiplicity* of *intrinsic mathematical expressions*, all of which identify *relative position* away from a ***pre-selected point*** *in space (Ref. Section 14)*.

Such concept is further characterized by introducing a *relativistic approach* which applies a *mobile origin* that is *perceived* to move about to *pre-selected points* upon an orthogonal *grid pattern*, thereby affording *different perspectives* with respect to such *stationary point*.

Now, *Parabolic* and *Generalized Cubic Function* ***coefficient structures*** are considered to be the very best possible candidates to represent respective *Quadratic* and *Cubic Function format* ***classifications*** because:

a. They limit *higher order expressions* to just one variable (or unknown), thereby promoting a *simplified mathematical analysis*, and
b. They allow for the greatest amount of *mathematical flexibility*.

However, *circles*, *ellipses*, *hyperbolas*, *Complex Quadratic Functions* and *Complex Cubic Functions* such as the one designated below also may qualify for subsequent treatment:

$$\alpha x^3+\beta x^2+\gamma x+\delta=y^2$$

Accordingly, ***selected*** *Parabolic* and *Generalized Cubic Function coefficient structures* are listed below:

$ax^2 + bx + c = y$ *(Ref. Section 14.1)*

$\alpha z^3 + \beta z^2 + \gamma z + \delta = y$ *(Ref. Section 14.2)*

The prospect of *realizing location* from a *singular point in space* is comparable to *pinpointing* an object by sonar, or wave reflection, whereby its *distance* away is easily calculated by assessing the *time* it takes for the wave to propagate to the object, multiplied by a predetermined *velocity* as it travels through a *known* medium.

For this study: *Triangulation*, which enables a position to be trigonometrically determined with respect to *two fixed points*, applies only when such second identified point is used to attribute an orientation for a *Cartesian Coordinate System* intended for use in a *Curve Mapping* analysis.

Various *travel route scenario* examples for the *Parabolic Curve Mapping process* are listed below:

a) Those which occur across its *root sets (Ref. Section 24 Related Problem Nos. 35 and 36)*;
b) Those which occur directly along a *Parabolic Curve (Ref. Section 24 Related Problem No. 38)*; and
c) Those which occur along any other *selected route*, such as over a *circular path* between *root sets (Ref. Section 24 Related Problem No. 40)*.

For each of these *Parabolic and Generalized Cubic Function coefficient structures, Curve Mapping methodology* consists of *(Ref. Abstract)*:

1) A *Singularity Proof* stating that all *family curves superimpose* onto a *parent curve* of *identical shape (Ref. Sections 14.1.1, and 14.2.1)*;
2) An accompanying *Algorithm* which reveals that a singular, stationary curve in space may be referred to by a *multiplicity* of *independent mathematical functions* which afford tracking or mapping capabilities *(Ref. Sections 14.1.2, and 14.2.2)*; and
3) An *Application subsection* which demonstrates precepts developed earlier by focusing upon certain detailed relationships that exist between *families of identically shaped curves (Ref. Sections 14.1.3, and 14.2.3)*.

<u>In conclusion</u>, an ***equation*** for a *fixed curve in space* is *not* absolute, but instead becomes *altered* depending upon an *observer's perspective*. Viewers who perceive such ***fixed curve*** from different vantage points can characterize it by alternate equations which also precisely depict it.

ITEM 6. Sub-element Theory Indicates how to Further Reduce Equation 1.

Over the years, some schools have maintained that *Equation 1* is irreducible, given that:

$$4\cos^3\theta = 3\cos\theta + \cos(3\theta) \qquad \textit{[Ref. Equation 1]}$$

In this regard, ***Equation sub-element theory*** shows how to reduce it from *cubic* to *quadratic form*; thereby refuting such assertions. This type of reduction can be achieved simply by letting (*Ref. Section 2.1.2*):

$$\tau = \cos(3\theta)$$

$$\lambda = \sin(3\varphi)\text{, and}$$

$$\sin\varphi = \frac{1}{2\cos\theta}$$

Accordingly, an *Equation 4* reduction of *Equation 1* results, as follows:

$$\cos^2\theta + (\frac{2\tau\lambda - 5}{6\lambda})\cos\theta - \frac{\tau}{2\lambda} = 0 \qquad \textit{[Ref. Equation 4]}$$

Next, *Equation 4* is confirmed for the specific case when 3θ is set equal to 60° (*Ref. Section 18.1*):

$$3\theta = 60^o$$

$$\theta = 60^o/3$$

$$= 20^o$$

$$\tau = \cos(3\theta) = \cos 60^o = 1/2$$

$$\sin\phi = \frac{1}{2\cos\theta} = \frac{1}{2\cos 20^o} = 0.532088886$$

$$\phi = 32.1467014^o$$

$$3\phi = 96.44010419^o$$

$$\lambda = \sin(3\phi) = 0.993689653$$

Then,

$$\cos^2\theta + (\frac{2\tau\lambda - 5}{6\lambda})\cos\theta - \frac{\tau}{2\lambda} = 0$$

$$(\cos 20^o)^2 + (\frac{2\tau\lambda - 5}{6\lambda})\cos 20^o - \frac{\tau}{2\lambda} = 0$$

$$(\cos 20^o)^2 + [\frac{2(1/2)(0.993689653) - 5}{6(0.993689653)}](\cos 20^o) - \frac{(1/2)}{2(0.993689653)} = 0$$

$$\cos^2 20^o - 0.671958683\cos 20^o - 0.251587605 = 0$$

$$0.883022221 - 0.631434616 - 0.251587605 = 0$$

$$0 = 0$$

In conclusion, *Equation 1* can *be reduced further*. Although *present day conjecture* is that such equation is ***irreducible***, reduction becomes precipitated simply by supplying applicable *irrational coefficients*, as determined by mathematical calculations presented above. As indicated, $\cos\theta$ may be determined directly via the *Euclidean mapping process* specified in Section 2.3, in order to enable its determination directly from a *straightedge* and *compass*.

ITEM 7. Sub-element Theory Algebraically Determines when Angle Trisectors can be Geometrically Constructed without having to Resort to Aforehand Knowledge of a Common Root Value Z_R.

Equation Sub-element Theory explains that *occurrences* of angle trisectors which can be *geometrically constructed*, although rather rare, can be identified by applying *Equation 51* cited below:

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3 + \gamma} \qquad \textit{[Ref. Equation 51]}$$

Criterion *(Ref. Section 19)*:

Angle trisection via geometric construction can be determined ***algebraically*** *as occurring* whenever *respective coefficient values* β, γ, and δ relating to a *second, independent Generalized Cubic Equation whose Sub-element R=1 can* be developed via *Equation 51* without having *aforehand knowledge* of its *common root value* z_R.

With respect to this above criterion, the ***root value*** z_R appearing in *Equation 51* is considered to be ***common*** to both of the following equations:

- The *3θ Cubic Equation*, hereinafter *also* referred to as a ***primary*** *Generalized Cubic Equation* of the form:

  $\tan^3\theta - 3\zeta\tan^2\theta - 3\tan\theta + \zeta - 0$ *[Ref. Equation 3]*

- A ***second,*** *independent Generalized Cubic Equation* whose *Sub-element R=1* of the form:

  $z^3 + \beta z^2 + \gamma z + \delta = 0$ *[Ref. Equation 32]*

  Where, *Sub-element R*=1 evidences conditions when*:*

  $z_R = \mathrm{R}\tan\theta = (1)\tan\theta = \tan\theta$ *[Ref. Section 10]*

Besides *Equation 51* cited above, other *yet to be defined equations* also show promise to surface additional potential occurrences of angles which can be *trisected* via singular compass and straightedge. In this regard, the possibility of linking a ***second,*** *independent Generalized Cubic Equation* whose *Sub-element R≠1* to the aforementioned *3θ Cubic Equation* through another common *root value*, either z_S or z_T could rather easily be accomplished, considering that:

$$z_R = \tan\theta_R = \mathrm{R}\tan\theta = (1)\tan\theta = \tan\theta$$

$$z_S = \tan\theta_S = \mathrm{S}\tan\theta \qquad \textit{[Ref. Section 11.6]}$$

$$z_T = \tan\theta_T = \mathrm{T}\tan\theta$$

7.1. *An Example of How to Algebraically Determine an Angle Trisector which can be Geometrically Constructed without having Aforehand Knowledge of Z_R.*

As an example, two occurrences where an angle trisector can be *geometrically constructed* by means of ***algebraic*** *determination* are identified below. This occurs by applying a *second, independent Generalized Cubic Equation (GCE)* that is devoid of its *second* and *third* terms such that *(Ref. Section 20)*:

$$\beta = \gamma = 0$$

Then, by substituting these coefficients into *Equation 51*:

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

$$= \frac{3\zeta(0-1)-4(0)}{3+0}$$

$$= -\zeta$$

Invoking *Equation 36 determines:*

$$\zeta = \frac{\delta-\beta}{1-\gamma} \qquad \textit{[Ref. Equation 36]}$$

$$\zeta = \frac{\delta-0}{1-0}$$

$$-z_R = \zeta = \delta$$

Hence, such *second, independent GCE* assumes the form:

$$z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$z_R{}^3 + (0)z^2 + (0)z - z_R = 0$$

$$z_R{}^2 = 1$$

$$z_R = R\tan\theta = (1)\tan\theta = \tan\theta = \sqrt{1}$$

$$\tan\theta_1; \tan\theta_2 = +1; -1$$

$$\theta_1; \theta_2 = \arctan(+1); \arctan(-1)$$

$$= 45^o; 135^o$$

$$3\theta_1; 3\theta_2 = 135^o; 405^o$$

$$= 135^o; 360^o + 45^o$$

$$= 135^o; 45^o$$

The calculations above indicate that *only* given angles of 135° and 45° can be ***trisected*** by sole use of *compass and straightedge* when considering *second, independent cubic equations* of $R = 1$ whose coefficients $\beta = \gamma = 0$.

This is because when $\zeta = \delta \neq \mp 1$ *second, independent Generalized Cubic Equations* of $R = 1$ other than the one specified above cannot meet the conditions set forth by the determined equation $z_R = -\zeta$. For example, given:

$$\zeta = \tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^o$$

$$\theta = 18.43494882^o$$

$$z_R = \tan\theta = 1/3$$

However, $z_R = 1/3 \neq -\zeta = -13/9$

Any astute reader should realize by now exacly why such above resuylt does not constitute a valid trisection solely by Euclidean means.

For the lesser accomplished reader, however, it furthermore is ventured that such finding does not commence from an angle of given magnitude 3θ; but instead algebraically determines that such angle must be either 135° or 45°.

Moreover, the contribution of any ***algebraic*** *determination* might be construed as an intervention of *aforehand knowledge*; thereby invalidating it as a valid avenue, or solution, to *geometrically construct* an angle whose magnitude amounts to exactly one-third the size of a given angle, solely by use of a straightedge and compass.

7.2. *An Example of How to Algebraically Determine an Angle Trisector which can't be Geometrically Constructed without having Aforehand Knowledge of Z_R.*

Appearing below is an example that further discusses such above ***common root value*** $z_R = 1/3$ for a 3θ value whose tangent amounts to a value of 13/9; thereby signifying a *rational root* which nevertheless still *cannot* be trisected solely by *Euclidean means (Ref. Section 19)*:

Where,

$$z_R = \tan\theta = \frac{1}{3}$$

$$\theta = 18.43494882^o$$

$$3\theta = 55.30484647^o$$

$$\zeta = \tan(3\theta) = \frac{13}{9}$$

By implementing *Equation 51*, it can be readily observed that the *modifying coefficients* β, γ, and δ inherent within a second, independent *Generalized Cubic Equation for* ***R=1*** <u>cannot</u> be determined; unless, of course, the actual value of such *common root value* z_R becomes introduced aforehand.

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma} \qquad \textit{[Ref Equation 51]}$$

$$z_R = \frac{3(13/9)(\gamma-1)-4\beta}{3+\gamma}$$

$$= \frac{(13)(\gamma-1)-12\beta}{3(3+\gamma)}$$

$$3z_R(3+\gamma) = (13)(\gamma-1) - 12\beta$$

Upon substituting $\gamma = -3$,

$$0 = (9\zeta)(-4) - 4(3)\beta$$

$$\beta = -3\zeta$$

The substitution described above becomes useless because such resulting coefficients revert back to those which apply to the *primary 3θ Cubic Equation*. Hence, no *second, independent Generalized Cubic Equation for R=1* can be generated when $\gamma = -3$.

For each value of γ other than -3 that is substituted into *Equation 51*, respective *coefficient values β*, and *δ* still cannot be readily ascertained, simply because the ***common root value*** z_R still remains unknown; thereby, foiling the prospect of identifying a trisector which might become geometrically determied.

ITEM 8. Sub-element Theory Proposes to Rectify Major Euclidean Limitation.

Equation Sub-element Theory proposes to *rectify* a major *Euclidean* limitation; that of being able to replicate *only* ***rationally-based lengths*** *(Ref. Section 22.2)*!

Because of this, *Euclidean practice* today looms as being somewhat ***incomplete***!

Rectification consists of enabling mathematicians to geometrically portray *cubic irrational lengths*, as well!

Whereas ***geometric construction*** is principally governed by a strict set of rules and regulations, it remains severely hampered simply because it has not been permitted to be analyzed beyond such *confining borders*!

To elaborate, *Euclidean practice* first should be defined in a manner conducive to everybody's liking. Naturally, in order to enhance it, it would be preferable to describe it with respect to itself!

Such statement might well assert, "*Euclidean practice* is exactly what it is stacked up to be; that is, exactly what it has been considered to accomplish since its very inception - No more, and no less"!

Stemming from such logic, another leading question is whether *Euclidean practice* is ***all*** that it *needs* to be.

In order to answer, a *reasonable* ***basis*** for *Euclidean practice* would have to be established -- one that seemingly would be considered to be acceptable to an entire mathematics community.

Such plausible basis unequivocally would hereinafter state,

- "*Euclidean renderings* should be what *can* be portrayed via *compass* and *straightedge* tools starting *only* from an arbitrarily assigned, or given length of unity -- No more, and no less" *(Ref. Section 9.1 Conclusion)*!

Accordingly, what appears to be at issue here is, not the *adequacy* of *Euclidean practice*, but its very ***relevancy.***

Such rectification proposes to amend present-day *Euclidean practice* by ***now*** allowing it to acknowledge ***overlapment*** as bonafide means of identifying *intersection (Ref. Item 4.1)*.

Hence, *Atacins* then could be legitimized as being *fully Euclidean* since it is comprised solely of *compasses* interconnected via straightedge, and regimented by *relative motion* and *geometric constraint* aspects entirely recognized as properties rudimentary to prevailing *Euclidean practice (Ref. Sections 22.3.1, 22.3.2* and *22.6)*.

Moreover, ***overlapment*** specifically pinpoints a *singular location* along the *longitudinal axis* of a straightedge which just so happens to *superimpose upon* or *coincide with* a *determinable point* in space; one which either may be stationary, or moving itself *(Ref. Section 22.3.3)*.

From the distant vantage point of Earth, such distinct *longitudinal axis*, once contemplated to exist outside of the realm of *Atacins*, may be perceived as a straight line of seemingly imperceptible width which becomes drawn, for example, through *Orien's Belt*. At the precise moment when it is observed to pass either directly in front of or behind a particular star, no matter how faint, ***overlapment*** occurs at the specific location where such straight line is viewed to cross, or intersect *with* the star.

Such process also may be likened to a *total eclipse* of the sun by the moon. At such time as this occurs, a straight line fictitiously can be drawn which is considered to *intersect*:

- The center of the moon
- The center of the sun
- The midway point between the viewer's eyes

Hence, ***overlapment*** coexists with *intersection*. They go hand-in-hand, whereby at times they even might be perceived as being *inextricably linked* or associated to one another. ruler ***(In the event of any conflict between this section and U.S. Patent No. 10994569 issued on 5/4/2021, the latter shall govern)***.

The only ***difference*** between them is that ***overlapment*** seeks to identify additional *intersection points* that previously either went undetected or otherwise were deliberately ignored during prior *geometric construction* exercises *(Ref. Section 22.3.3)*.

Had *Euclid* and his contemporaries been advised that *cubic irrational lengths* actually could be depicted solely from a unique arrangement of *compasses* interconnected via *straightedge*, such capability most definitely would have been incorporated into their practice long ago.

Such esoteric notion of intersection points, considered to be germane to ***geometry***, nevertheless remains *fundamental* to generally accepted *Euclidean practice* as well. Hence, the newly fashioned property of ***overlapment***, because it also locates such points, should be categorized under this very same, overall *Euclidean umbrella*.

Such approach, not only would render overall *geometric operations* more complete in the future *(Ref. Section 22.2)*, but ultimately avail the portrayal of *cubic irrational lengths* alongside *rationally-based* ones -- thereby equipping mankind with a very important, new capability!

The prospect of incorporating *cubic irrational* ***length*** *portrayals* into formerly established *Euclidean practice* without violating, detracting from, or otherwise conflicting with its precepts *theoretically* would entail *(Ref. Section 22.4)*:

- Using only *Euclidean compass* and *straightedge* instruments in a manner entirely consistent with all of the rules and regulations applied during Euclid's day
- Treating ***cubic irrational length geometrically formed*** *depiction* in exactly the same manner as ***rationally-based geometric construction***; whereby both become determinable entirely from a given length of unity *(Ref. Section 9.1)*
- Acknowledging the *process* of obtaining ***geometrically formed*** depictions as a new Euclidean ***enhancement***; one which remains completely independent, or is distinguished entirely apart from the presently accepted *Euclidean process* of ***geometric construction***

ITEM 9. Sub-element Theory Renders Former Approximation Techniques Obsolete.

Atacins enables *cubic irrational lengths* to be portrayed as ***exact measurements*** *(Ref. Section 22.6.4)*.

Since *cubic irrational lengths* describe *decimal sequences* which are considered to *continue on indefinitely*, instead of repeating themselves, oftentimes approximation techniques, like the one described below, have been administered to replicate them:

> *Dividing up a given length of unity into ten equal portions (Ref. Figure 45), and then into hundredths (Ref. Figure 46), and so on, until such desired cubic irrational length becomes amply gauged via ruler.*

Exact ***geometrically formed*** depictions do a much better job than such *approximation* techniques. Hence, in many cases, mathematicians should consider the latter as becoming *obsolete.*

ITEM 10. Sub-element Theory Maps Pi.

With the advent of *Atacins*, it now becomes possible to map $\pi = 3.14159265358979323846264338327 9...$ in terms of *rational lengths* and *trigonometric aspects* of the circle *(Ref. Related Problem Number 48)*. Rational length significance can be to whatever number of decimal places becomes desired.

ITEM 11. Sub-element Theory Establishes a Complete Cubic Resolution Transform.

All *Generalized Cubic Equations* can be resolved by means of a ***complete*** *Cubic Resolution Transform (Ref. Section 13.3).*

It consists of an *overall*, or *universal* ***cubic resolution capability*** that amplifies or expands upon *fragmented*, or *partially presented prior state-of-the-art techniques (Ref. Section 13.3.6).*

The *Cubic Resolution Transform (CRT)* serves to unify such *aforementioned theories* into a more powerful, comprehensive *algorithm* that exhibits the following unique attributes:

11.1. CRT Enables Direct Resolution of All Generalized Cubic Equations.

The *CRT* ***directly resolves*** all *Generalized Cubic Equations*, *regardless* of what format they may appear in *(Ref. Sections 13.3.3 thru 13.3.5* and *Related Problems 26 thru 33).* In stark contrast, *prevalent present day resolutions* are limited in the sense that they can operate *only* upon *cubic formats* which are *devoid of* their second order terms (*Ref. Section 13.3.6*). Accordingly, they require that *most given Cubic Equations first* become subjected to the additional step of undergoing a *transformation* before resolution can be accomplished (*Ref. Section 13.3.6*).

In conclusion: From a ***number theory*** *standpoint*, such *previously developed algorithms* more appropriately *now* should be *categorized* as *sub-classifications* to this *newly proposed*, *universal* **Cubic Resolution Transform.**

11.2. CRT Entails a Definitive Geometry .

The CRT entails, or is predicated upon, a definitive geometry (Ref. *Figure 11*) which now may be applied to such limited, present day cubic resolutions in order to allow them to be represented *geometrically (Ref. Section 13.3.6).*

11.3. CRT Specifies Newly Identified Coefficient Driven Property ψ=cos(6ω) .

The *CRT* contains a newly identified term $\psi = \cos(6\varpi)$ which is ***coefficient driven***, or *fully distinguishable* by mere *manipulation* of respective *Generalized Cubic Equation coefficient* values *(Ref. Section 13.3.3).* It joins a *distinguished list* of other ***coefficient driven*** terms:

- $x_1 ; x_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ *[The* ***well-known*** *Quadratic Formula]*
- $\zeta = \tan(3\theta) = \dfrac{\delta - \beta}{1 - \gamma}$ *[Ref. Equation 36]*
- $\psi = \cos(6\varpi) = \dfrac{9\gamma\beta - 2\beta^3 - 27\delta}{2(\beta^2 - 3\gamma)^{\frac{3}{2}}}$ *[As derived from Equation 42]*

$\psi = \cos(6\varpi)$ can be determined for any specific *Generalized Cubic Equation coefficient structure*. For example, when $\beta = 0$:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$\alpha z^3 + \gamma z + \delta = 0$$

Then,

$$\psi = \cos(6\varpi) = \frac{9\gamma\beta - 2\beta^3 - 27\delta}{2(\beta^2 - 3\gamma)^{\frac{3}{2}}}$$

$$= \frac{9\gamma(0) - 2(0)^3 - 27\delta}{2[(0)^2 - 3\gamma]^{\frac{3}{2}}}$$

$$= \frac{\delta}{2(-\frac{\gamma}{3})^{\frac{3}{2}}} \qquad \textit{[Ref. Equation 41]}$$

11.3.1. Direct Computation of cos(2ω) and its Associated Cubic Equation Root z_R .

The term $\psi = \cos(6\varpi)$ allows for computation of the value of $\cos(2\varpi)$ directly from it, and thereafter enables final determination of such unknown *Cubic Equation root* z_R as follows:

$$z_R = -\frac{1}{3}[\beta - 2\sqrt{\beta^2 - 3\gamma}\cos(2\varpi)] \qquad \textit{[As derived from Equation 42]}$$

Such algebraic overcomes an *inability*, in most cases, to *trisect 6ω angles via Euclidean geometric construction (Ref. Figure 11 and Section 20)*.

11.3.2. Determination for Imaginary Cubic Equation Roots .

The term $\psi = \cos(6\varpi)$ readily deciphers whether a given *Generalized Cubic Equation* contains *imaginary root sets (Ref. Sections 13.3.1 and 13.3.2, 13.3.4 and 13.3.5)*, such that,

If $-1 \le \psi \le +1$, *three real roots* exist; if not, an imaginary set applies.

Figure 39 depicts a set of possible *curve scenarios* where, as shown, only the *middle curve* renders *three real roots (Ref. Section 15.3)*.

11.4. CRT Adds Criteria for the Trigonometric Solution of the Cubic Equation.

The *CRT* affords *three possible format selections* listed as follows, only one of whose respective, *coefficient sign conventions* match those specified in any *given*, or *postulated Generalized Cubic Equation* devoid of its second term. Such match-up *must* be conducted *prior* to performing resolution via the ***well-known*** *Trigonometric Solution of the Cubic Equation (Ref. Section 13.3.6)*:

- $4\cos^3\theta - 3\cos\theta - \cos(3\theta) = 0$ *[Ref. Equation 1]*
- $4\sin^3\theta - 3\sin\theta + \sin(3\theta) = 0$ *[Ref. Equation 2]*
- $4\sinh^3 x + 3\sinh x - \sinh(3x) = 0$ *[Ref. Section 13.3.6]*

ITEM 12. Sub-element Theory Resolves Cubic Equations when Sub-element R=1.

Generalized Cubic Equations (GCE's) which contain ***Sub-elements*** of unity can be algebraically resolved.

They're easily distinguished because the sums of the *coefficients* of their associated *Characteristic Cubic Equations* always equal zero. This is demonstrated as follows, where *(Ref. Section 13.1)*:

$$AR^3 + BR^2 + CR + D = 0 \qquad \text{[Ref Equation 31]}$$

$$AS^3 + BS^2 + CS + D = 0$$

$$AT^3 + BT^2 + CT + D = 0$$

$$A(1)^3 + B(1)^2 + C(1) + D = 0$$

$$A + B + C + D = 0$$

This applies to *Equation 3*, a specific *GCE* that exhibits a root of $z_R = R\tan\theta = (1)\tan\theta = \tan\theta$ *(Ref. Sections 2.4.3 and 10)*.

Then, for the specific case when $\zeta = \tan(3\theta) = \tan 65.90515745^o = \sqrt{5}$:

$\tan^3\theta = 3\tan\theta - \tan(3\theta)(1 - 3\tan^2\theta)$	*[Ref. Equation 3]*

$$z^3 - 3\zeta z^2 + -3z + \zeta = 0$$

$$z^3 + (B\tan\theta)z^2 + (C\tan^2\theta)z + D\tan^3\theta = 0 \qquad \text{[Ref. Section 11.3]}$$

$$B = \frac{-3\zeta}{\tan\theta} = \frac{-3\sqrt{5}}{\tan(65.90515745^o/3)} = \frac{-3\sqrt{5}}{\tan 21.96838582^o} = \frac{-3\sqrt{5}}{0.403384527} = -16.62979979$$

$$C = \frac{-3}{\tan^2\theta} = \frac{-3}{\tan^2 21.96838582^o} = \frac{-3}{(0.403384527)^2} = -18.43668279$$

$$D = \frac{\zeta}{\tan^3\theta} = \frac{\sqrt{5}}{\tan^3 21.96838582^o} = \frac{\sqrt{5}}{(0.403384527)^3} = 34.06648253$$

Then: $A + B + C + D = 1 - 16.62979979 - 18.43668279 + 34.06648253 = 0$

Therefore,

$$z_R = \tan\theta = \tan(3\theta/3) = \tan(65.90515745^o/3) = \tan 21.96838582^o = 0.403384527$$

Check,

$$\tan^3\theta = 3\tan\theta - \tan(3\theta)(1 - 3\tan^2\theta)$$

$$(0.403384527)^3 = 3(0.403384527) - \sqrt{5}[1 - 3(0.403384527)^2]$$

$$0.065638358 = 1.210153583 - 2.236067978(1 - 0.488157231)$$

$$= 1.210153583 - 2.236067978(0.511842768)$$

$$= 1.210153583 - 1.144515225$$

$$= 1.210153583 - 1.144515225 \qquad \textit{Q.E.D.}$$

Next, the inherent RST technology can be determined as follows:

$$\mathrm{S} = -\frac{1}{2}[(\mathrm{B}+1)) \pm \sqrt{4\mathrm{D} + (\mathrm{B}+1)^2}]$$

$$= -\frac{1}{2}[(-16.62979979 + 1) \pm \sqrt{4(34.06648253) + (-16.62979979 + 1)^2}]$$

$$= -\frac{1}{2}(-15.62979979 \pm \sqrt{136.2659301 + 244.2906415})$$

$$= -\frac{1}{2}(-15.62979979 \pm \sqrt{380.5565716})$$

$$= -\frac{1}{2}(-15.62979979 \pm 19.50785923)$$

$$= -\frac{1}{2}(3.878059436;\ -35.13765902)$$

$$= -1.939029718;\ 17.56882951$$

Such that,

$$B = -(R + S + T)$$
$$= -(1 + S + T)$$

$$D = -RST$$
$$= -(1)ST$$
$$= -ST$$

$$A + B + C + D = 0$$
$$C = -(A + B + D)$$
$$= -[1 - (1 + S + T) - ST]$$
$$= -(1 - 1 - S - T - ST)$$
$$= -(-S - T - ST)$$
$$= S + T + ST$$
$$= S + T(1 + S)$$

$$-18.43668279 = (-1.939029718;\ 17.56882951) + T(-0.939029718;\ 18.56882951)$$
$$-18.43668279 + (1.939029718; -17.56882951) = (-0.939029718;\ 18.5688295)T$$
$$(-16.49765307; -36.0055123) = (-0.939029718;\ 18.5688295)T$$
$$(-16.49765307; -36.0055123)/(-0.939029718;\ 18.5688295) = T$$
$$17.56882956; -1.939029721 = T$$

$$B = -(R + S + T)$$
$$= -[1 + (-1.939029718;\ 17.56882951) + (17.56882956; -1.939029721)]$$
$$= -[(-0.939029718;\ 18.56882951) + (17.56882956; -1.939029721)]$$
$$= -16.62979979 \qquad \text{Q.E.D.} \ \surd$$

Whereby,

$$D = -ST$$
$$= -(-1.939029718;\ 17.56882951)(17.56882956; -1.939029721)$$
$$= -(-1.939029718;\ 17.56882951)(17.56882956; -1.939029721)$$
$$= 34.06648253 \qquad \text{Q.E.D.} \ \surd$$

ITEM 13. Sub-element Theory Showcases an Equation which can Dispense RST Terminology.

Equation Sub-element Theory showcases a novel *missing link transform*, hereinafter referred to as the ***Unified Cubic Trigonometric Reduction Equation*** *(Ref. Equation 29)*. It serves as a *direct conduit* whereby *RST Terminology* embedded within *Cubic Equations* can be *dispensed* into *reduced Quadratic Equations*. Because of such *linkage*, resulting *lower order Quadratic Equation reductions* thereafter house vital *higher order Cubic Equation* information *(Ref. Section 10)*.

ITEM 14. Sub-element Theory Introduces RST Spreads.

RST Spreads represent an *amalgamation* of *root set spacings* that apply to any *given Generalized Cubic Function (GCF)*. They accrue as the z-axis becomes displaced *vertically* with respect to such curve, now considered to be *stationary (Ref. Section 15)*.

Hence, they depict an assortment of relative *root set* spans which exist along such *GCF* as it becomes viewed horizontally from *different elevations*.

The *algebraic format* for the *GCF* is established simply by replacing the zero appearing on the right-hand side of the Generalized *Cubic Equation 32* by the variable *y* as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \text{[Ref. Equation 32]}$$

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = y$$

$$\alpha z^3 + \beta z^2 + \gamma z + (\delta - y) = 0$$

As such, ***RST Spreads*** depict an assortment of *root sets*, each of which applies to a specific *y* value that becomes specified and then entered into the above equation.

14.1. RST Spreads Designate the very Realms of Three Dimensional Space.

RST Spreads may be distinguished, or perceived, as *deviation* from a ***Three Dimensional Space Norm*** where:

- Such ***norm***, or ***benchmark*** is to be represented as the *function* for *Equation 22* as follows, selected because *RST Spreads* become *useful* when they are categorized with respect to *3θ Cubic Functions* which they modify, or belong to:

  $$z^3 - 3\zeta z^2 - 3z + \zeta = y \quad \text{[Ref. Equation 22]}$$

  Where,

  $$z_R = \tan\theta_R = R\tan\theta = \tan\theta$$

  $$z_S = \tan\theta_S = S\tan\theta = \tan(\theta + 120^o)$$

  $$z_T = \tan\theta_T = T\tan\theta = \tan(\theta + 240^o)$$

  $$\theta_R + \theta_S + \theta_T = 3\theta + 360^o = 3\theta$$

  Such ***norm***, hereinafter is to be referred to as the *3θ Cubic Tangent Function*, or just the ***3θ Cubic Function***.

  As indicated above, ***RST Spreads*** track *RST Terminology* in the form of accrued, respective *root set* values divided by tan θ.

- A ***Three Dimensional Space Norm*** is to be represented via the ***volumetric expletive RST***, otherwise expressed as the negative of *coefficient 'D'*, specified as follows in the *Characteristic Cubic Equation*:

  $$AR^3 + BR^2 + CR + D = 0 \quad \text{[Ref. Equation 31]}$$

  $$AS^3 + BS^2 + CS + D = 0$$

  $$AT^3 + BT^2 + CT + D = 0$$

  Where,

  $$A = 1$$

  $$B = -(R + S + T)$$

  $$C = RS + RT + ST$$

  $$D = -RST$$

As such, ***RST Spreads*** designate *the very realms of three dimensional space which 3θ Cubic Functions* occupy *(Ref. Section 15.2)*.

RST Spreads for the *norm* are constructed by reconstituting the *3θ Cubic Function* into *equation form*, and then solving for its roots *(Ref. Section 15.2)*. Based upon this, *Figure 29* depicts an associated ***RST Spread*** for the *norm* when $\zeta = \tan(3\theta) = \tan 60^{\circ} = \sqrt{3}$.

14.2. RST Spreads Allow for an Exact Matching of Cubic Function Curve Shapes.

It is now possible to identify a *3θ Cubic Function* which assumes the same exact shape as any *given Generalized Cubic Curve (Ref. Section 15.4)*.

Such affinity applies, as well, to their:

a) *Associated 3θ displaced family curves*;
b) *Displaced family curves*; and
c) *Parent curves*.

More specifically, all above listed *curve shapes* can be shown to be virtually identical except for the fact that they are simply *translated*, or moved to different locations about the origin *(Ref. Section 15.1)*.

Accordingly, all above listed curve shapes must contain the same exact coterie of *RST Spreads* within them.

Hence ***RST Spreads*** inherent within ***3θ Cubic Functions*** characterize the *root structures* for <u>all</u> ***Generalized Cubic Functions***.

Such association of *RST Spreads* enables these *seemingly diverse cubic functions* now to be portrayed, collated, disseminated, and thereafter compiled with respect to one another in an effective manner *(Ref. Figure 41* and *Table 33)*.

One specific example relates just how a given *RST Spread* inherent to one set of *cubic functions* can be traced, or charted, to another *(Ref. Section 15.1)*.

14.3. RST Spreads Exhibit an Interchangeability Attribute.

When a horizontal line becomes constructed somewhere between *points* y_A and y_B on a *3θ Cubic Function (Ref. Figure 19 and Section 14.2.2)*, ***three ordinates*** drawn through locations where it intersects such curve always make contact with respective *R*, *S*, and *T curves* at only *three distinct elevations (Ref. Figure 29 and Figure 30* and *Table 29)*.

This manifestation is known as an ***interchangeability attribute*** which maintains that *volumetric expletives*, commensurate with the product *RST*, always *remain constant*.

Such ***interchangeability attribute*** serves to *identify* an underlying intrinsic quality which otherwise remains hidden within *Cubic Functions*; one which normally would escape the purview of most mathematicians.

A related *volumetric expletive* is rendered in *Figure 31*. It graphically depicts a $(z)(z_1)(z_2) = z_f z_1 z_2$ curve to be of the exact same shape as its associated *3θ Cubic Function*, but riding a distance of ζ below it. Hence, its ordinate or *y value*, as a *volumetric expletive*, is equal to the $z_f z_1 z_2$ product.

14.4. RST Spreads Chart Imaginary Number Thresholds .

Notice that S and T Curves shown in *Figure 29* become bounded by *vertical projections* which emanate from points where respective y_A and y_B horizontal lines intersect the *3θ Cubic Function*.

For such *3θ Cubic Function*, the *vertical projections* designate thresholds of exactly where the *real number* root set ends and the *imaginary number* root set begins.

This is easily understood when viewing *Figure 39*. It shows three identically shaped curves with only the middle one being located such that the abscissa resides within its y_A and y_B horizontal projections. As such, this curve exhibits three respective roots; whereas, the other two exhibit only one root each.

Since *cubic roots* are defined as respective *z locations* where *cubic curves* actually intersect with the z-axis, the term ***imaginary*** retains absolutely no ***real*** context, other than that it represents non-existent entities which allow *cubic equation coefficients* to become mathematically expressed in terms of one *real root* and two other *fictitious numbers*.

Moreover, as the ***norm*** or *3θ Cubic Function* assumes different values of ζ, its S and T curve thresholds move.

Accordingly, *real root regions* become different for each *S and T Curve* represented *(Ref. Figure 35, Figure 36* and *Figure 37)*. *Table 31* represents the basis for such plot by charting *RST Curves* with respect to *z*. For each *3θ Cubic Curve*, it indicates the spans over which the *S and T Curves* remain real and locates exactly where they become imaginary. Therein, respective *R*, *S*, and *T* values are determined as follows:

$$R = \frac{z_f}{\tan\theta'}$$

$$S = \frac{z_1}{\tan\theta'}$$

$$T = \frac{z_2}{\tan\theta'}$$

Within their respective *real root regions*, *S and T Curves* associated with the *3θ Cubic Curve Sets* expressed in *Figure 32* are depicted in *Figure 34* where,

S represents the Lower portion, and T pertains to the upper portion of each curve. *S and T Curves* are joined, or connected, at respective left-most and right-most portions of each curve, respectively.

As an illustrative example, an ***RST Spread*** is developed for the associated *function* of *the 3θ Cosine Cubic Equation* $z^3 - (3/4)z - \tau/4 = y$ *(Ref. Equation 1* and *Section 15.3)*.

Figure 40, generated through calculations posed in *Table 32*, portrays an ***RST Spread*** superimposed over the *3θ Cosine Cubic Function* whose S and T Terminology exists only within the range $-1 \le \cos\theta = z = z_f \le +1$.

14.5. RST Spreads Distinguish 3θ Cubic Function Variability .

Figure 32 represents sixteen *3θ Cubic Functions* which exhibit various *arbitrarily selected* 3θ values. Such mapping displays the *variability* evidenced by the *3θ Cubic Function* as it undergoes change in its *fundamental property*, ζ.

Figure 33 portrays associated *R values* for the various *3θ Cubic Curves* presented in *Figure 32.* For any ordinate selected, representing a *constant value* for R, *Figure 33* illustrates just how much *shape change* occurs to *Figure 32 3θ Cubic Functions* while moving to the right, or increasing in z value; where, *Cubic Curve shape* itself may be viewed as another ultimate property.

ITEM 15. Sub-element Theory Establishes Linear Relationship between ζ=tan(3θ) and tan θ.

Based <u>solely</u> upon a *specific mathematical manipulation* of *Characteristic Cubic Equation* coefficient values (*Ref. Equation 31*), a **new** <u>significant</u> *linear* relationship between the tan θ and its associated $\zeta = \tan(3\theta)$ function is established as follows:

$$\tan\theta = -(\frac{J}{F})\zeta$$ *[Ref. Equation 48]*

Where,

$F = 2[3D - B]$

$J = 3(B + C) - (D + 1) \pm G$

Such that,

$G = \pm\sqrt{9(B^2 + C^2) + D^2 + 14BC - 6BD + 6CD + 1 + 6B - 6C - 34D}$

ITEM 16. Sub-element Theory Develops J-Function Cubic Expression.

Equation Sub-element Theory develops a *J-function Cubic Expression* cited below which enables quick determination of its unknown *root sets (Ref. Section 17.2)*:

$$J^3 + (3F)J^2 - 3(\frac{F}{\xi})^2 J - F(\frac{F}{\xi})^2 = 0$$ *[Ref. Equation 49]*

Such determination applies whenever a given *Cubic Equation* exhibits the above enumerated *coefficient groupings*.

Equation 49 is analogous with the *3θ Cubic Equation (Ref. Section 17.3)*.

Equation 49 conveniently can be applied in consonance with *Equation 48* to make sense out of, or analyze, a rather dissonant set of what before appeared to be inscrutable mathematical intangibles *(Ref. Section 17.4)*.

ITEM 17. Sub-element Theory Founds Normalization Transformation.

Equation Sub-element Theory founds the *Normalization Transformation of Parabolic Functions* which verifies when families or sets of *parabolic curves* are *identically shaped* by *graphically superimposing* them upon one another.

This is demonstrated for *eighteen parabolic curves* as portrayed in *Figure 15*, all of the form $ax^2 + bx + c = y$ with their respective *coefficients*, as tabulated in *Table 18*, reflecting the following three specific relationships:

1) $a = 3$

2) $b = a(\frac{1 - 3\tan^2\theta}{2\tan\theta})$

3) $c = \frac{a}{2}(\tan^2\theta - 1)$

Figure 15 applies over the range $-15 \le x \le +15$. Respective ordinate, or y-axis values represent the sum of left-hand terms for the *Parabolic Function* $ax^2 + bx + c = y$. *Table 19* tabulates such plot points.

The *Normalization Transformation of Parabolic Functions* translates these *eighteen parabolic functions*, as portrayed in *Figure 15*, from their respective x_M and y_M *low point locations* to the origin; thereby, illustrating that all such curves superimpose, or overlap upon their family curve $3x^2 = y$ *(Ref. Figure 16)*. *Table 20* tabulates such plot points. As mathematically indicated, all y values in each row of *Table 20* always equal a resulting product of $3x^2$.

The Normalization Transformation of Parabolic Functions is of the form shown below *(Ref. Section 14.1.3)*:

$$a(x + x_M)^2 + b(x + x_M) + (c - y_M) = y$$

Essential algorithmic relationships for such charted *curves* are characterized as $x_M = -b/(2a)$ and $y_M = ax_M{}^2 + bx_M + c$.

Figure 17 represents an example application by disclosing four identically shaped *parabolic curves* such that the low point of one of them, their $3x^2 = y$ family curve, resides upon the origin.

Figure 18 renders a second plot which illustrates the respective positions of the three other identical curves once they have been translated via the *Normalization Transformation of Parabolic Functions*. As indicated, they are demonstrated to precisely align, or coincide with such $3x^2 = y$ *Parent Curve*.

ITEM 18. Sub-element Theory Unveils the Trisector Equation Generator.

Equation Sub-element Theory unveils *Equation 52, a Trisector Equation Generator* for the particular case when:

$z_R = -\beta / 3$.

Its derivation shown below, is ***inspired*** by the realization that when respective *Generalized Cubic Equation coefficients* $\alpha = 1$ and $\delta = \beta^3 / 27$, the ***cube root*** term ***adds to zero*** for the following circumstance:

$$z_R = R\tan\theta = \frac{-\beta + \sqrt[3]{\beta^3 - 27\alpha^2\delta}}{3\alpha} \qquad \textit{[Ref. Section 13.2 Formula]}$$

$$= \frac{-\beta + \sqrt[3]{\beta^3 - 27(1)^2\delta}}{3(1)}$$

$$= \frac{-\beta + \sqrt[3]{\beta^3 - 27(\beta^3 / 27)}}{3}$$

$$= \frac{-\beta + \sqrt[3]{\beta^3 - \beta^3}}{3}$$

$$= \frac{-\beta}{3}$$

Moreover, since this above circumstance applies when $\beta^2 = 3\alpha\gamma = 3(1)\gamma = 3\gamma$ *(Ref. Section 13.2)*:

Then,

$$\gamma = \beta^2 / 3$$

Accordingly

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 + \beta z^2 + (\beta^2 / 3)z + \beta^3 / 27 = 0$$

$$(z + \beta / 3)^3 = 0$$

$$(z_R + \beta / 3)(z_S + \beta / 3)(z_T + \beta / 3) = 0$$

Therefore,

$$z_R + \beta / 3 = 0$$

$$z_R = -\beta / 3 \qquad \textit{Q.E.D.}$$

$$z_S + \beta / 3 = 0$$

$$z_S = -\beta / 3$$

$$z_T + \beta / 3 = 0$$

$$z_T = -\beta / 3$$

Notice that for this above circumstance, it is not necessary to take a ***cube root***.

Furthermore, the value of the *coefficient* β can be either *rationally-based*, or *cubic irrational*.

The *geometric construction* aspect of this analysis becomes rather rudimentary since it consists simply of *geometrically* dividing any value of the *coefficient β*, as specified in a *Generalized Cubic Equation* whose other *coefficients* are in the following proportions, into *three equal portions* in order to determine its associated *root* z_R:

$$\alpha = 1$$
$$\gamma = \beta^2/3$$
$$\delta = \beta^3/27$$

Naturally since it is of far greater interest to derive an ***algorithm*** which instead determines equation types from given, or known values of $\zeta = \tan(3\theta)$ where their associated ***cube root*** terms also *add out to zero*, the above methodology is further applied in order to derive *Equation 52* as follows*:*

Where,

$$z_R = R\tan\theta = (1)\tan\theta = \tan\theta$$
$$\alpha {z_R}^3 + \beta {z_R}^2 + \gamma z_R + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$
$$(1)\tan^3\theta + \beta\tan^2\theta + \gamma\tan\theta + \delta = 0$$

Such that,

$$\zeta = \frac{\delta - \beta}{1-\gamma} \qquad \textit{[Ref. Equation 36]}$$
$$\zeta(1-\gamma) + \beta = \delta$$
$$\zeta(1-\frac{\beta^2}{3}) + \beta = \delta$$

Substitution below renders:

$$\tan^3\theta + \beta\tan^2\theta + \gamma\tan\theta + \delta = 0$$
$$\tan^3\theta + \beta\tan^2\theta + (\frac{\beta^2}{3})\tan\theta + [\zeta(1-\frac{\beta^2}{3}) + \beta] = 0$$
$$\frac{\beta^2}{3}(\tan\theta - \zeta) + \beta(1+\tan^2\theta) + \tan^3\theta + \zeta = 0$$
$$\beta^2 + [\frac{3(1+\tan^2\theta)}{(\tan\theta-\zeta)}]\beta + (3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)} = 0$$

Completing the square gives:

$$\beta^2 + [\frac{3(1+\tan^2\theta)}{(\tan\theta-\zeta)}]\beta + [\frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}]^2 + (3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)} = \frac{9(1+2\tan^2\theta+\tan^4\theta)}{4(\tan\theta-\zeta)^2}$$

Then,

$$[\beta + \frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}]^2 = \frac{9(1+2\tan^2\theta+\tan^4\theta)}{4(\tan\theta-\zeta)^2} - (3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)}(\frac{4}{4})(\frac{\tan\theta-\zeta}{\tan\theta-\zeta})$$
$$\beta + \frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}] = [\frac{1}{2(\tan\theta-\zeta)}]\sqrt{9(1+2\tan^2\theta+\tan^4\theta) - 12(\tan^3\theta+\zeta)(\tan\theta-\zeta)}$$

Equation 52 $$\beta = \frac{-3(1+\tan^2\theta) \pm \sqrt{9(1+2\tan^2\theta+\tan^4\theta) - 12(\tan^4\theta - \zeta\tan^3\theta + \zeta\tan\theta - \zeta^2)}}{2(\tan\theta-\zeta)}$$

Therefore, for any postulated real value of $\zeta = \tan(3\theta)$ and its associated, calculated value $z_R = R\tan\theta = (1)\tan\theta = \tan\theta$, the coefficients β, $\gamma = \beta^2/3$ and $\delta = \beta^3/27$ can be calculated in order to describe a *Generalized Cubic Equation* whose root $z_R = -\beta/3$.

ITEM 19. Sub-element Theory Discredits Attempts to Trisect by means of Geometrically Constructing Cube Roots.

In the past various methods were resorted to which relied upon attempts to *geometrically construct* ***cube roots***.

The association that ***Equation sub-element theory*** bears upon such *cube root* interpretations is presented below. In many cases, algebraic interpretations are supplied, thereby becoming disqualified as methods which could be used to accomplish *Euclidean trisection (Ref. Section 21)*.

19.1. Geometrically Constructing Cube Roots is Synonomous with performing Euclidean Trisection, and therefore cannot be achieved solely by Euclidean Means.

With regard to the factor $\cos(2\omega)$, as contained in the variable ℓ of the *Cubic Resolution Transform (CRT) presented* below, an association with ***cube roots*** can be established as follows *(Ref. Section 13.3)*:

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right) f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0 \qquad \textit{[Ref. Equation 38]}$$

Such that

$$\ell = 2f\cos(2\omega) \qquad \textit{[Ref. Figure 11]}$$

Where the formula for a *Binomial Expansion* of the *cube* of the *polynomial* $A \pm B$ is as follows:

$$(A \pm B)^3 = A^3 \pm 3A^2B + 3AB^2 \pm B^3$$

For the specific circumstance when:

$$A = \cos(2\omega)$$

$$B = i\sin(2\omega)$$

$$\begin{aligned}(A \pm B)^3 &= [\cos(2\omega)]^3 \pm 3[\cos(2\omega)]^2[i\sin(2\omega)] + 3[\cos(2\omega)][i\sin(2\omega)]^2 \pm [i\sin(2\omega)]^3 \\ &= \cos^3(2\omega) \pm 3[1-\sin^2(2\omega)][i\sin(2\omega)] - 3[\cos(2\omega)][1-\cos^2(2\omega)] \mp i\sin^3(2\omega) \\ &= [4\cos^3(2\omega) - 3\cos(2\omega)] \pm i[3\sin(2\omega) - 4\sin^3(2\omega)] \\ &= \cos(6\omega) \pm i\sin(6\omega)\end{aligned}$$

Taking the *cube root* of each side affords:

$$A + B = \cos(2\omega) + i\sin(2\omega) = \sqrt[3]{\cos(6\omega) + i\sin(6\omega)}$$

$$A - B = \cos(2\omega) - i\sin(2\omega) = \sqrt[3]{\cos(6\omega) - i\sin(6\omega)}$$

Such that by summing the two above equations,

$$2\cos(2\omega) = \sqrt[3]{\cos(6\omega) + i\sin(6\omega)} + \sqrt[3]{\cos(6\omega) \quad i\sin(6\omega)}$$

Now, upon letting ψ represent $\cos(6\omega)$, the following equality can be established,

$$\cos^2(6\omega) + \sin^2(6\omega) = 1$$

$$\psi^2 + \sin^2(6\omega) = 1$$

$$\sin(6\omega) = \sqrt{1-\psi^2}$$

Then, by substituting this result into the equation above, it can be shown that,

$$
\begin{aligned}
2\cos(2\omega) &= \sqrt[3]{\psi + i\sqrt{1 - \psi^2}} + \sqrt[3]{\psi - i\sqrt{1 - \psi^2}} \\
&= \sqrt[3]{\psi + i\sqrt{(-1)(\psi^2 - 1)}} + \sqrt[3]{\psi - i\sqrt{(-1)(\psi^2 - 1)}} \\
&= \sqrt[3]{\psi + i^2\sqrt{\psi^2 - 1)}} + \sqrt[3]{\psi - i^2\sqrt{\psi^2 - 1)}} \\
&= \sqrt[3]{\psi - \sqrt{\psi^2 - 1)}} + \sqrt[3]{\psi + \sqrt{\psi^2 - 1)}}
\end{aligned}
$$

Since real values for ψ exist within the range from -1 to +1, then the radical $\sqrt{\psi^2 - 1}$ must be *imaginary* or equal to zero. Hence, except for such latter case, each of the terms appearing under the two *cube root radicals* indicated above must be *complex numbers*. Now, since taking the *cube root* of a complex number is synonymous with representing its trisector in a Cartesian Coordinate System, it would appear to be impossible to *geometrically construct* it solely by *Euclidean means*.

19.2. Showing how Cube Roots can be Eliminated through Algebraic Manipulation.

Except for certain *very rare instances (Ref. Section 20)*, an *unknown quantity z* may be represented as the *negative* ***cube root*** of the summation of *second*, *third* and *fourth terms* of a given *Generalized Cubic Equation* for $\alpha = 1$ that becomes mathematically reorganized as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$
\begin{aligned}
z^3 + \beta z^2 + \gamma z + \delta &= 0 \\
z^3 &= -\beta z^2 - \gamma z - \delta \\
&= (-1)(\beta z^2 + \gamma z + \delta) \\
&= (-1)^3(\beta z^2 + \gamma z + \delta) \\
z &= -\sqrt[3]{\beta z^2 + \gamma z + \delta}
\end{aligned}
$$

Since such *2nd and 3rd terms* include the *unknown root, z*, its value is ***required aforehand*** in order to determine the value of the left-hand side of the above equation. Hence, such algebraic relationship cannot contribute towards attempting to trisect an angle solely by Euclidean means *(Ref. Section 19)*.

19.2.1 For Rational Values of Z_R and ζ when R=1 and β=0.

Interposing rational values of $z_R = \tan\theta$ and $\zeta = \tan(3\theta)$ into the *3θ Cubic Equation* enables results to be obtained which thereafter could be geometrically constructed, as based upon such input. For example, when $z_R = 1/3$ *(Ref. Section 19 Example)*:

$$z_R^{\ 3} - 3\zeta z_R^{\ 2} - 3z_R + \zeta = 0 \quad \textit{[3θ Cubic Equation]}$$

$$(1/3)^3 - 3\zeta(1/3)^2 - 3(1/3) + \zeta = 0$$

$$1/27 + \zeta(1 - 1/3) - 1 = 0$$

$$\zeta(18/27) = 26/27$$

$$\zeta = 13/9$$

A *second, independent Generalized Cubic Equation (GCE)* for $R = 1$ and $\beta = 0$ can be determined as:

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3 + \gamma} \quad \textit{[Ref. Equation 51]}$$

$$1/3 = \frac{3(13/9)(\gamma - 1) - 4(0)}{3 + \gamma}$$

$$(1/3)(3 + \gamma) = (13/3)(\gamma - 1)$$

$$3 + \gamma = 13\gamma - 13$$

$$16 = 12\gamma$$

$$4/3 = \gamma$$

Hence, the two above determined equations can be combined in order to be *resolved* simultaneously via the *Quadratic Formula*, or the ***geometric construction*** *Mapping Process* presented in *Section 2.3*, as follows:

$$z_R^{\ 3} - 3\zeta z_R^{\ 2} - 3z_R + \zeta = 0$$

$$z_R^{\ 3} = 3\zeta z_R^{\ 2} + 3z_R - \zeta$$

For $\alpha = 1$

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$z^3 + \beta z^2 + \gamma z + \delta = 0$$

Via substitution from above into the *Quadratic Formula*:

$$[3\zeta z_R{}^2 + 3z_R - \zeta] + \beta z_R{}^2 + \gamma z_R + \delta = 0$$

$$(3\zeta + \beta) z_R{}^2 + (3 + \gamma) z_R + (\delta - \zeta) = 0$$

$$z_R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ *[Ref. Quadratic Formula]*

$$z_R = \frac{-(3+\gamma) \pm \sqrt{(3+\gamma)^2 - 4(3\zeta + \beta)(\delta - \zeta)}}{2(3\zeta + \beta)}$$

$$= \frac{-(3 + 4/3) \pm \sqrt{(13/3)^2 - 4(13/3 + 0)[\zeta(1-\gamma) + \beta - \zeta]}}{2(13/3 + 0)}$$

$$= \frac{-(13/3) \pm \sqrt{(169/9) + 4(13/3)(\zeta\gamma - \beta)]}}{2(13/3)}$$

$$= \frac{-(13/3) \pm \sqrt{(169/9) + 4(13/3)(13/9)(4/3)}}{26/3}$$

$$= \frac{-(13/3) \pm (1/3)\sqrt{169 + 4(13/3)(13)(4/3)}}{26/3}$$

$$= \frac{-13 \pm 13\sqrt{(9+16)/9}}{26}$$

$$= \frac{-1 \pm 5/3}{2}$$

$$= 1/3; -4/3$$

<u>Accordingly</u>:

- From a *given angle* $3\theta = 55.30484647^o$, $\zeta = \tan(3\theta) = 13/9$ <u>can</u> be ***geometrically constructed***
- From the synthesis of such *two equations*, a common root $z_R = \tan\theta = 1/3$ can be ***geometrically constructed*** using the Quadratic Equation expressed above via the mapping process stipulated in *Section 2.3*
- From such ***geometrically constructed*** length of $z_R = \tan\theta = 1/3$, an *angle* θ then can be ***geometrically constructed*** which is equal to 18.43494882^o, or exactly 1/3 the magnitude of such *given angle* $3\theta = 55.30484647^o$. Since such geometric construction relies upon the results of an algebraic analysis as *aforehand knowledge*, such process does <u>not</u> qualify as a valid ***Euclidean*** trisection

Above, notice that it is not necessary to extract a ***cube root*** in order to *algebraically* determine such solution.

19.2.2 For β=γ=0.

An associated analysis begins by examining the *Generalized Cubic Equation* for conditions when $\alpha = 1$ as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 = -(\beta z^2 + \gamma z + \delta)$$

$$z = -\sqrt[3]{\beta z^2 + \gamma z + \delta}$$

Notice above that in order to calculate a ***root z***, it first becomes necessary to extract the ***cube root*** of a value which is comprised of *multiples* and *mathematical combinations* of such unknown *quantity*.

However, this doesn't apply when $\beta = \gamma = 0$ as follows:

$$z^3 + (0)z^2 + (0)z + \delta = 0$$

$$z^3 + \delta = 0 \qquad \textit{[Ref. Section 13.5]}$$

Where,

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

$$= \frac{\delta - 0}{1 - 0}$$

$$= \delta$$

Via substitution:

$$z_R{}^3 + \delta = 0$$

$$z_R{}^3 + \zeta = 0$$

$$(R \tan\theta)^3 + \zeta = 0$$

When $R = 1$, the above equation then relates $\tan\theta$ to $\zeta = \tan(3\theta)$ where,

- $\zeta = \tan(3\theta)$ is a value which can be *geometrically constructed* from any ***given angle 3θ***
- $\tan\theta$ is a value from which ***trisected angle θ*** can be *geometrically constructed*

Under such conditions,

$$\tan^3\theta + \zeta = 0$$

Via further substitution of *Equation 3*:

$$\tan^3\theta + \frac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta} = 0$$

$$\tan^2\theta + \frac{(3-\tan^2\theta)}{1-3\tan^2\theta} = 0$$

$$\tan^2\theta(1-3\tan^2\theta) + 3 - \tan^2\theta = 0$$

$$-3\tan^4\theta + 3 = 0$$

$$-\tan^4\theta + 1 = 0$$

$$1 = \tan^4\theta$$

$$\pm 1 = \tan^2\theta$$

$$\pm 1; i = \tan\theta$$

$$\pm 45^o = \theta$$

$$45^o; 315^o = \theta$$

$$135^o; 945^o = 3\theta$$

$$135^o; 225^o = 3\theta$$

$$\mp 1 = \zeta$$

Accordingly,

$$\tan^3\theta + \zeta = 0$$

$$\tan^3\theta \mp 1 = 0$$

$$\tan\theta = \pm\sqrt[3]{1}$$

$$\tan\theta = \pm 1$$

Since the ***cube root*** of *unity* is defined as *unity*, an algebraic solution becomes afforded without having to extract such ***cube root***.

This above *finding* is independently confirmed by *Equation 51* which applies because $z_R = R\tan\theta = (1)\tan\theta = \tan\theta$ as follows:

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

$$= \frac{3\zeta(0-1)-4(0)}{3+0}$$

$$= -\zeta \qquad \textit{[Ref. Section 20.1]}$$

$$= z_R{}^3 \qquad \textit{[Since } z_R{}^3 + \zeta = 0 \textit{ above]}$$

$$1 = z_R{}^2$$

Taking the *square root* produces values for z_R as follows:

$$\sqrt{1} = z_R$$

$$\pm 1 = z_R$$ *[Ref. Section 20.1]*

$$= \tan\theta$$

$$\arctan(\pm 1) = \theta$$

$$45^o; 135^o = \theta$$

$$135^o; 45^o = 3\theta$$

$$\tan 135^o; \tan 45^o = \tan(3\theta)$$

$$-1; +1 = \zeta$$

Check,

$$z_R{}^3 + \zeta = 0 \qquad z_R{}^3 + \zeta = 0$$

$$z_R{}^3 - 1 = 0 \qquad z_R{}^3 + 1 = 0$$

$$1^3 - 1 = 0 \qquad (-1)^3 + 1 = 0$$

$$1 - 1 = 0 \qquad -1 + 1 = 0$$

$$0 = 0 \qquad 0 = 0$$

As such, the two specifically determined *Generalized Cubic Equations*, $z_R{}^3 = \pm 1$, do not require ***cube roots*** to be *geometrically constructed* because they can be reduced to respective *Quadratic Equations* as demonstrated above.

19.2.3 For Circumstances when Generalized Cubic Equations Exhibit Coefficients in Prescribed Ratios.

One good example of first *interpreting*, and thereafter *geometrically operating* upon the *coefficient structures* of given *cubic algebraic equations* pertains to a ***cubic root*** which, in fact, is equal to a fraction of a coefficient which appears in a *Generalized Cubic Equation*.

For purposes of illustration, for:

$$\beta = -3z_R$$

$$-\frac{\beta}{3} = z_R$$

$$0 = z_R + \frac{\beta}{3}$$

$$0 = (z_R + \frac{\beta}{3})^3$$

$$= z_R{}^3 + 3(\beta/3)z_R{}^2 + 3(\beta/3)^2 z_R + (\beta/3)^3$$

$$= z_R{}^3 + \beta z_R{}^2 + (\beta^2/3)z_R + \beta^3/27$$

Such that,

$$0 = \alpha z^3 + \beta z^2 + \gamma z + \delta \qquad \textit{[Ref. Equation 32]}$$

Matching *like coefficients* renders:

$$\alpha = 1$$
$$\gamma = \beta^2/3$$
$$\delta = \beta^3/27$$

As such, a *Generalized Cubic Equation* whose ***coefficients*** appear in the *respective proportions* afforded below contains a *root* equal to $z_R = -\beta/3$:

$$z_R{}^3 + \beta z_R{}^2 + (\beta^2/3)z_R + \beta^3/27 = 0$$

Notice that for this above case, the value of the *coefficient β* is either *rationally-based*, or *cubic irrational.*

The *geometric construction* aspect of this analysis becomes rudimentary since it consists simply of *geometric*ally dividing any given value of β into three equal portions in order to determine the value of its associated *root* z_R.

Moreover, since $\beta^2 = 3\alpha\gamma = 3(1)\gamma = 3\gamma$, the following equation also applies *(Ref. Section 13.2)*:

$$z_R = R\tan\theta = \frac{-\beta + \sqrt[3]{\beta^3 - 27\alpha^2\delta}}{3\alpha}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - 27(1)^2\delta}}{3(1)}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - 27(\beta^3/27)}}{3}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - \beta^3}}{3}$$
$$= \frac{-\beta}{3}$$

As indicated above, the ***cube root*** term always adds out to zero when making use of such *Generalized Cubic Equation* ***format***.

Unfortunately, this *above analysis* represents *little more* than determining equations for any ***prescribed root*** z_R whose *coefficient β can be acted upon via* ***geometric construction*** for purposes of <u>again</u> identifying or producing such ***given root***.

Three other *Cubic Equations* of the above *format* are determined below through a *simplified process*. One exhibits a *rational cubic root*, another contains a *cubic root* comprised of a *square root quantity* that can be *geometrically constructed* via the mapping process specified in *Section 2.3*, and another expresses an *cubic irrational cubic root* as follows:

For

$$z_R = \tan\theta = 1/5$$

$$\beta = -3z_R = -3/5$$

$$\gamma = \beta^2/3 = 3/25$$

$$\delta = \beta^3/27 = \gamma\beta/9 = -1/125$$

For

$$z_R = \tan\theta = 3+\sqrt{7}$$

$$\beta = -3z_R = -3(3+\sqrt{7})$$

$$\gamma = \beta^2/3 = 3(16+6\sqrt{7})$$

$$\delta = \beta^3/27 = \gamma\beta/9 = -1(90+34\sqrt{7})$$

For

$$z_R = \tan\theta = \tan 20^o = 0.363970234$$

$$\beta = -3z_R = -1.091910703$$

$$\gamma = \beta^2/3 = 0.397422994$$

$$\delta = \beta^3/27 = \gamma\beta/9 = -0.048216713$$

Check,

$$z^3 - \frac{3}{5}z^2 + (\frac{3}{25})z - \frac{1}{125} = 0$$

$$(\frac{1}{5})^3 - \frac{3}{5}(\frac{1}{5})^2 + (\frac{3}{25})(\frac{1}{5}) - \frac{1}{125} = 0$$

$$1-3+3-1=0$$

$$0=0$$

$$z^3 - 3(3+\sqrt{7})z^2 + 3(16+6\sqrt{7})z - (90+34\sqrt{7}) = 0$$

$$(3+\sqrt{7})^3 - 3(3+\sqrt{7})(3+\sqrt{7})^2 + 3(16+6\sqrt{7})(3+\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(27+27\sqrt{7}+63+7\sqrt{7}) - 3(3+\sqrt{7})(16+6\sqrt{7}) + 3(16+6\sqrt{7})(3+\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(90+34\sqrt{7}) + (3-3)(3+\sqrt{7})(16+6\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(90+34\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$0=0$$

$$z^3 - 1.091910703z^2 + 0.397422994z - 0.048216713 = 0$$

$$(0.363970234)^3 - 1.091910703(0.363970234)^2 + 0.397422994(0.363970234) - 0.048216713 = 0$$

$$0.048216713 - 0.14465014 + 0.14465014 - 0.048216713 = 0$$

$$0=0$$

For the first of these above determined Cubic Equations, roots may be determined ***linearly*** via the expression *posed in Equation 51 as follows:*

For $z^3 - \frac{3}{5}z^2 + (\frac{3}{25})z - \frac{1}{125} = 0$

Where,

$$\zeta = \frac{\delta - \beta}{1-\gamma} \qquad \text{[Ref. Equation 36]}$$

$$= \frac{-\frac{1}{125} + \frac{3}{5}(\frac{25}{25})}{\frac{125}{125} - \frac{3}{25}(\frac{5}{5})}$$

$$= \frac{74}{110}$$

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3+\gamma} \qquad \text{[Ref. Equation 51]}$$

$$= \frac{3(\frac{74}{110})[\frac{3}{25}(\frac{5}{5}) - (\frac{125}{125})] - 4(-\frac{3}{5})(\frac{25}{25})}{3(\frac{125}{125}) + \frac{3}{25}(\frac{5}{5})}$$

$$= \frac{3(\frac{74}{110})(-110) + 300}{375+15}$$

$$= \frac{78}{390}$$

$$= \frac{1}{5} \qquad \text{Q.E.D.}$$

19.2.4 For Applications of the Trisector Equation Generator.

As presented below, *Equation 52* can be used to determine <u>equation types</u> from given, or known values of $\zeta = \tan(3\theta)$ such that their associated ***cube root*** terms also *add out to zero.*

$$\beta = [\frac{1}{2(\tan\theta - \zeta)}][-3(1+\tan^2\theta) \pm \sqrt{9 + 12\zeta^2 - 12\zeta\tan\theta + 18\tan^2\theta + 12\zeta\tan^3\theta - 3\tan^4\theta}]$$

Based upon the derivation of such equation, it turns out that for any *postulated real value* of $\zeta = \tan(3\theta)$ and its *associated, calculated value* $z_R = R\tan\theta = (1)\tan\theta = \tan\theta$, the *coefficients* β, $\gamma = \beta^2/3$, and $\delta = \zeta(1 - \frac{\beta^2}{3}) + \beta$ can be calculated in order to describe a *Generalized Cubic Equation* whose *root* is $z_R = -\beta/3$.

Now, for the specific case when it is intended that:

$$\zeta = \tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^{o}$$

$$\theta = 18.43494882^{o}$$

$$z_R = \tan\theta = 1/3$$

Equation 52 yields:

$$\beta = [\frac{1}{2(\tan\theta - \zeta)}][-3(1+\tan^2\theta) \pm \sqrt{9+12\zeta^2 - 12\zeta\tan\theta + 18\tan^2\theta + 12\zeta\tan^3\theta - 3\tan^4\theta}]$$

$$= [\frac{1}{2(-10/9)}][-3(10/9) \pm \sqrt{(729+2028)/81 - 468/81 + 162/81 + 52/81 - 3/81}]$$

$$= [\frac{-9}{20}][-\frac{30}{9} \pm \frac{1}{9}\sqrt{(729+2028) - 468 + 162 + 52 - 3}]$$

$$= [\frac{-1}{20}][-30 \pm \sqrt{2757 - 468 + 162 + 49}]$$

$$= \frac{3}{2} \mp \frac{1}{20}\sqrt{2500}$$

$$= \frac{3 \mp 5}{2}$$

$$= -1; +4$$

$$\gamma = \frac{\beta^2}{3}$$

$$= \frac{1}{3}; \frac{16}{3}$$

$$\delta = \zeta(1-\gamma) + \beta$$

$$= \frac{13}{9}(1-\frac{1}{3}) - 1; \frac{13}{9}(1-\frac{16}{3}) + 4$$

$$= \frac{13}{9}(\frac{2}{3}) - \frac{27}{27}; \frac{13}{9}(-\frac{13}{3}) + 4(\frac{27}{27})$$

$$= \frac{1}{27}; -\frac{61}{27}$$

Hence, such above determined *coefficients* generate the following pair of *Generalized Cubic Equations*:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 - z^2 + \frac{1}{3}z - \frac{1}{27} = 0$$

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

Check,

For

$$z^3 - z^2 + \frac{1}{3}z - \frac{1}{27} = 0$$

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3 + \gamma}$$ *[Ref. Equation 51]*

$$= \frac{3(\frac{13}{9})[\frac{1}{3} - 1(\frac{3}{3})] - 4(-1)(\frac{9}{9})}{3(\frac{9}{9}) + \frac{1}{3}(\frac{3}{3})}$$

$$= \frac{(\frac{13}{3})(\frac{-2}{3}) + (\frac{36}{9})}{\frac{27}{9} + \frac{3}{9}}$$

$$= \frac{10}{30}$$

$$= \frac{1}{3}$$

For

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3 + \gamma}$$

$$= \frac{3(\frac{13}{9})[\frac{16}{3} - 1(\frac{3}{3})] - 4(4)(\frac{9}{9})}{3(\frac{9}{9}) + \frac{16}{3}(\frac{3}{3})}$$

$$= \frac{(\frac{13}{3})(\frac{13}{3}) - 4(4)(\frac{9}{9})}{\frac{27+48}{9}}$$

$$= \frac{25}{75}$$

$$= \frac{1}{3}$$

Also:

$$\beta^2 = 3\alpha\gamma$$
$$= 3(1)(1/3)$$
$$\beta = \pm\sqrt{1}$$
$$\beta_2 = -1$$

$$\beta^2 = 3\alpha\gamma$$
$$= 3(1)(16/3)$$
$$\beta = \pm\sqrt{16}$$
$$\beta_1 = +4$$

So,

$$z_R = \frac{-\beta_2 + \sqrt[3]{\beta_2^{\,3} - 27\alpha^2\delta}}{3\alpha}$$

$$= \frac{+1 + \sqrt[3]{(-1)^3 - 27(+1)^2(-\frac{1}{27})}}{3}$$

$$= \frac{+1 + \sqrt[3]{-1+1}}{3}$$

$$= \frac{1+0}{3}$$

$$= \frac{1}{3}$$

$$z_R = \frac{-\beta_1 + \sqrt[3]{\beta_1^{\,3} - 27\alpha^2\delta}}{3\alpha}$$

$$= \frac{-4 + \sqrt[3]{(4)^3 - 27(1)^2(-\frac{61}{27})}}{3}$$

$$= \frac{-4 + \sqrt[3]{125}}{3}$$

$$= \frac{-4+5}{3}$$

$$= \frac{1}{3}$$

And:

$$z^3 - z^2 + (1/3)z - 1/27 = 0$$

$$(\frac{1}{3})^3 - (\frac{1}{3})^2(\frac{3}{3}) + \frac{1}{3}(\frac{1}{3})(\frac{3}{3}) - \frac{1}{27} = 0$$

$$(1 - 3 + 3 - 1)/27 = 0$$

$$0 = 0$$

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

$$(\frac{1}{3})^3 + 4(\frac{1}{3})^2(\frac{3}{3}) + \frac{16}{3}(\frac{1}{3})(\frac{3}{3}) - \frac{61}{27} = 0$$

$$\frac{1 + 12 + 48 - 61}{27} = 0$$

$$0 = 0$$

Now with regard to these newly determined equations, The ***common root*** $z_R = 1/3$ for the first *given Cubic Equation* above can be *geometrically constructed* without having to take a *cube root* since such ***cube root term*** *adds out to zero*.

Moreover, such first *given Cubic Equation*, as cited above, contains $z_R = 1/3 = -\beta/3$ as a root; thereby representing the tangent of the *trisected angle θ*, the latter of which then could be *geometrically constructed* very easily.

With regards to the *second above given Cubic Equation*, $z^3 + 4z^2 + (16/3)z - 61/27 = 0$, its *associated root* z_R *can be geometrically constructed* from its *given coefficients* via application of *Equation 51*, as shown above. Hence, in this particular case, it also is not necessary to obtain a ***cube root*** via *geometric construction*.

Therefore, a *given angle* of $3\theta = 55.30484647^o$ can be ***divided*** *into three equal angles* of $\theta = 55.30484647^o / 3 = 18.43494882^o$ each by means of a *geometric construction* which utilizes nothing more than a *straightedge* and *compass* when ***applying*** the *coefficients* and *respective formats* expressed in *either* of the above determined *Cubic Equations*.

In conclusion, ***Generalized Cubic Equation formats*** exhibiting a ***sub-element*** *of* $R = 1$ contain a *root* of $z_R = \tan\theta$ with respect to their characteristic values of $\zeta = \tan(3\theta)$ such that,

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

Accordingly, such values z_R and ζ can be determined by a *geometric construction* which employs *only straightedge and compass* instruments that operate solely upon *various inherent coefficients* resident within these *formats*.

Since an ***angle of 3θ*** can be ***geometrically constructed*** from a *given* value of $\zeta = \tan(3\theta)$, and since an ***angle of θ*** also can be ***geometrically constructed*** from such previously *algebraically determined* value of $z_R = \tan\theta$, trisection can be *achieved* through *geometric manipulation* of such *inherent coefficients*.

This does not constitute a bonafide *Euclidean Trisection* event since such ***Generalized Cubic Equation formats*** exist merely as algebraic *transformations* that constitute ***aforehand knowledge*** of such *desirable root structures* in the first place *(Ref. Section 19)*.

ITEM 20. Sub-element Theory Monitors and Substantiates its Own Adequacy.

Equation Sub-element Theory monitors, as well as, substantiates its very own adequacy by actually resolving *sample problems* which have been introduced into various *treatise analysis* and *application sections*. The problems serve to validate intended hypothesis performance.

Moreover, *Sections 13* and *18* present numerous problem examples as well.

For example, resolution of any given *Generalized Cubic Equation* is best achieved by first interpreting its coefficient structure, and then administering the easiest to apply resolution method.

Such *interpretation* and resulting method of resolution consists either of:

a) Determining whether any inherent ***R, S, and T sub-element values*** equal unity *(Ref. Section 13.1)*;
b) Discerning whether their coefficients relate to one another in a certain prescribed manner *(Ref. Sections 13.2, 13.4 and 13.5)*; or
c) Ascertaining the root structure via *Cubic Resolution Transform* since it applies to all Generalized Cubic Equations *(Ref. Section 13.3)*.

Lastly, *Section 24* incorporates almost 150 additional pages of *related problems* that are presented in the same sequence as respective *supporting theories* are afforded in the treatise itself. This locates related problems that apply to specific hypotheses all in one particular area.

ITEM 21. Sub-element Theory Mathematically Quantifies Moving Waves.

Equation Sub-element Theory mathematically equates *moving waves* to a *series of curves* that infinitesimally change shape over time *(Ref. Figure 60)*.

RST Spread *variability* renderings, such as those portrayed in *Figure 33*, may be used to chart such small shape changes; thereby, enabling their correlation with respect to *instantaneous forces* which either may be applied during such wave action, or that become imposed as the result of such movement *(Ref. Section 23)*.

ITEM 22. Sub-element Theory Enlists Complex Quadratic Equation Formats.

Equation Sub-element Theory enlists *Complex Quadratic Equation formats* to mathematically arrange *multiple quantities* of unknowns.

They suitably depict *unknown* physical and thermodynamic properties which fundamentally are considered to change value over time.

Section 3.4.1 cites well-known *Complex Quadratic Equations* that are evidenced within the field of Physics.

Such equations, in turn, become mathematically operated upon in order to produce new expressions *(Ref. Section 3.4.2)*.

Such reformatting, or transformation, enables them to be specifically adapted for purposes of resolving physics characterizations which otherwise might remain very difficult to quantify *(Ref. Related Problem Number 9; Section 24)*.

Equation Sub-element Theory recruits other *equation formats* (alongside *Equation 30, Equation 36* and *Equation 48* addressed earlier) which all also possess the capability to link *trigonometric values* of an angle to those of one-third its size.

These consist of the following formats *(Ref. Section 18)*:

- An Equation 1 Reduction
- Equations resulting when $z_R - 1/\tan(3\theta) = -1/\zeta$
- *Complex Quadratic Equations* for the Angle Trisector Triangle
- Equations Emulated by the Cosine Circle

ITEM 23. Sub-element Theory Relates Elliptical Relationships.

Equation Sub-element Theory relates *Cubic* and *Sixth Order Elliptical Relationships* (Ref. Equations 44 and 45) to the geometry posed in *Figure 12*.

ITEM 24. Sub-element Theory Linearizes the Cube.

Three equations derived for purposes of *Linearizing the Cubic* are presented as follows:

- $\frac{1}{\sin\theta} = \frac{\tau}{\eta}(\frac{1}{\cos\theta}) + \frac{2}{\eta}$
- $\frac{1}{\sin(2\theta)} = -\frac{\tau}{\eta}[\frac{1}{\cos(2\theta)}] + \frac{2}{\eta}$
- $\frac{1}{\sin(4\theta)} = \frac{\tau}{\eta}[\frac{1}{\cos(4\theta)}] - \frac{2}{\eta}$

Such *Linearizations*, or *Cubic Equation reductions* into *Linear form*, are depicted in *Figure 9*.

This process may be viewed as actually skipping over *quadratic representations* entirely, or transforming from *Cubic Equations* directly into associated Linear reductions.

ITEM 25. Sub-element Theory Charts Physics Law of Reflection.

Transmitted rays whose *angles of incidence* and *reflection* are equal now can be precisely charted via *Atacins* without having to apply a protractor *(Ref. Figure 54)*.