THE PRINCIPLES OF

EQUATION SUB-ELEMENT THEORY

ABSTRACT

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This treatise *formally establishes* the ***Principles of Equation Sub-elements*** – being a *headlong excursion* into the topsy-turvy preoccupation of *classifying mathematical equation formats*.

Equation sub-elements, hereinafter deemed RST terminology, reveal just how Quadratic and Cubic Equations behave with respect to one another.

They operate from behind the scenes, governing equation interaction through a network of strict rules.

RST terminology acts to associate coefficient structures evident within algebraic equation formats to their very root sets; thereby enabling them to be directly solved through the use of newly presented formulas.

RST sub-elements appear as respective ***factors*** serving to *characterize Generalized Cubic Equation root set values* z_R, z_S, and z_T during specific circumstances when such equation's *coefficient* α is set equal to unity as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$ *Generalized Cubic Equation [Ref. Equation 32]*

$$z^3 + \beta z^2 + \gamma z + \delta = 0$$

Where,

$$z_R = R \tan\theta = \tan\theta_R$$

$$z_S = S \tan\theta = \tan\theta_S$$ *[Ref. Section 10]*

$$z_T = T \tan\theta = \tan\theta_T$$

As indicated directly above, *RST terminology* also relates the tangent of an *angle* θ to respective tangents of three *root set characteristic angles*, hereinafter denoted as θ_R, θ_S, *and* θ_T, the sum of which equals 3θ degrees as follows:

$$\theta_R + \theta_S + \theta_T = 3\theta$$ *[Ref. Section 10]*

Accordingly, *Quadratic* and *Cubic Equations now* can be linked via *trigonometric sets of* θ *and* 3θ that exist *within* existing *root sets* and constituent *coefficient structures*. For example,

- The *Generalized Cubic Equation* is of *universal significance* because it accounts for all *Cubic Equation* possibilities where 'z' appears as the *only* unknown quantity. Therein, *tan* θ presents itself as a *factor* to *all three* roots z_R, z_S, and z_T (as indicated above); whereas $\xi = \tan(3\theta)$ manifests itself as an *Overall Equation Characteristic Value* that readily can be determined via manipulation of *Equation 32 coefficients* in accordance with *Equation 36* shown below:

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Section 36]*

- The *Simplified Unified Cubic Trigonometric Reduction Equation (SUCTRE) (Ref. Equation 30)* reiterated below exhibits tan θ as its principal *quadratic unknown*; whereby, $\xi = \tan(3\theta)$, is a factor contained within both its first and third term coefficients:

$$\zeta[C + 3D]\tan^2\theta - [B - 3D]\tan\theta - \zeta(D + 1) = 0$$

Accordingly, for each value of ζ identified in a specific *Generalized Cubic Equation*, there exists an *associated SUCTRE* which features identical tan θ and $\xi = \tan(3\theta)$ aspects.

SECTION 6

Equation Sub-element categorization *begins* by creating a ***hierarchy chart*** *(Ref. Table 10)* which exhibits the following attributes:

- It *categorizes equations* and *functions* by section, where
  - **Section 2** depicts *Fundamental Information*
  - **Section 3** depicts *Complex Quadratic Equations*
  - **Section 4** depicts *Complex Quadratic Functions*
  - **Section 5** depicts *Cubic Equations* and *Functions*

  Such that,

  Complex Quadratic Equations *relate* various combinations of first and second order *multiple unknown quantities* (such as 'x_1', 'x_2', etc) to their *coefficients (Ref. Section 2.2)*.

  Such appellation is meant to differentiate them from regular, or normal *Quadratic Equations* which *relate* first and second order combinations of just a *singular* unknown quantity; in this case, 'x', to various *coefficients*.

  Complex Quadratic Equations allow for *special monitoring* of *multiple unknowns* where each can become individually *interrogated*. This is similar to the manner in which *partial differential equations* may be used to *identify* specific values for typical *thermodynamic properties* such as pressure, volume, and density, by *acting upon one variable at a time* while ascribing distinct values to such other unknowns.

  Such concept also extends to *Complex Linear Equations* which contain multiple unknowns that only are expressed linearly.

- It expresses *parent lineage*, or *paths of development*, which, by quick glance, help to determine various similarities and differences that exist between the equation types expressed above
- It identifies *distinguishing details* that exist between respective equations, in order to rapidly segregate those which possess identical z_1 or ζ values in common.

Section 6.1 cites *certain* ***similarities*** which independent equations derived from the <u>same</u> *parent equation* bear in common. For example, *Complex Quadratic Functions* referred to in *Figures 4 thru 6* exhibit *only* two roots each; and these entail all of the possibilities for identifying two out of three roots of the *Figure 7 Cubic Function* plot.

Section 6.2 identifies *certain* ***differences*** which otherwise exist; whereby, *Complex Quadratic Equations can* exhibit roots *not* contained in their associated *Cubic Equations.*

SECTION 7

Three equations derived for purposes of *Linearizing the Cubic* are presented as follows:

- $\frac{1}{\sin\theta} = \frac{\tau}{\eta}(\frac{1}{\cos\theta}) + \frac{2}{\eta}$
- $\frac{1}{\sin(2\theta)} = -\frac{\tau}{\eta}[\frac{1}{\cos(2\theta)}] + \frac{2}{\eta}$
- $\frac{1}{\sin(4\theta)} = \frac{\tau}{\eta}[\frac{1}{\cos(4\theta)}] - \frac{2}{\eta}$

Such *Linearizations*, or *Cubic Equation reductions* into *linear form*, are depicted in *Figure 9*.

This process may be viewed as actually skipping over *quadratic representations* entirely, or transforming from *Cubic Equations* directly into associated linear reductions.

SECTION 8

Identities encompass *indeterminate equations* whose formats defy mathematical resolution.

Such definition applies *even* to *Cubic Equation formats* which express only *singular unknown quantities* such as those enumerated in *Table 12*.

Each of these above equations is considered to be extraordinary in that it manifests *only* a *singular unknown* but, nevertheless, still defies mathematical resolution!

Reductions of *Cubic* or even *Higher Order Equations* can be achieved by substituting respective right-hand *lower order terms* of equations *presented in Table 12* for left-hand *cubic equivalencies* appearing in other equations.

For example, with regard to *Quartic Equations*, applicable *cubic expressions expressed in Table 12* need to be substituted for twice, in order to reduce into *Quadratic Equation* format.

In some identities, each and every included equation *coefficient equates to zero (Ref. Section 8.3)*.

In others, *numerical summations of respective terms* on each side of the equation may equate. *Equality* is still maintained because left-hand side and right-hand terms sum to zero *(Ref. Section 8.4)*.

Hence, such *identities* cannot provide quantitative indication of unknown numerical value. However, they can *validate* that mathematical calculations conducted during the *reduction process* were performed *correctly*!

A *constituent geometry* for generating cosine related *identities* in presented in *Section 8.1*.

Equation 27 presents its *Complex Quadratic Relationship*, while *Equation 28* depicts its associated *Quartic Relationship (Ref. Section 8.2)*.

Identities can become reconciled when *particular unknowns* become subject to a further *mathematical scrutiny* which enables their determination. Generally, such values become ascertained through other equations which are established independent from the identity needing to the resolved.

> For example, the *infinite number of solutions* that apply to the ***general*** *Cubic Equation* $z^3 - 3\zeta z^2 - 3z + \zeta = 0$ may be reduced to just *three* once a *particular value* for ζ becomes selected, or determined elsewhere, thereby permitting its entry.
>
> As ζ is assigned or accorded a *particular value*, such as $\tan 60^o = \sqrt{3}$, the ***general*** *Cubic Equation* thereby becomes transformed into $z^3 - 3\sqrt{3}z^2 - 3z + \sqrt{3} = 0$. Hence, the resulting *singular unknown z* then assumes just *three* determinable values of:
>
> $$z_1 = \tan\theta = \tan 20^o$$
>
> $$z_2 = \tan(\theta + 120^o) = \tan 140^o$$
>
> $$z_3 = \tan(\theta + 240^o) = \tan 260^o$$

On a grander scale, *identities can be* completely resolved by a process hereinafter referred to as ***mathematical closure*** *(Ref. Section 8.5)*.

For *Complex Quadratic Equation 27*, which consists of just *two variables*, it is pointed out that a *complete resolution*, or *mathematical closure*, becomes achieved only after ***all possible values*** that could be assigned to *one variable* determine the infinite number of associated values for the other.

As such, equations which harbor a *multiplicity of solutions too numerous* to be assessed mathematically, now can be resolved simply by specifying *which* particular *root sets* are to undergo further treatment.

SECTION 9

In order to *advance Number Theory state-of-the-art*, an attempt is made to explain the very existence for varying *equation formats*, and the reason why *diversity* exists between them.

To this end, a comparison is conducted between *Quadratic and Cubic Equation formats* which reveals that:

a) Each exhibits a *mathematical structure* that actually is quite different in nature from the other;
b) Each functions in a *diverse* manner; and
c) Each exists for a unique reason!

Section 9.1 asserts that *all* mathematical numbers can be categorized either as *rationally-based* or *cubic irrational*, where:

Rationally-based numbers consist of:

a. *All rational values; and*
b. *Quadratic irrational values* such as $17\sqrt{35\sqrt{7/1025}}$ which are comprised of *all* lengths that can be ***geometrically constructed*** via *Pythagorean Theorem* either from *solely rational lengths* in concert with an infinite variety of mathematical combinations of other *purely rational lengths*, or from their results.

Cubic irrational numbers consist of other *irrational values* and account for <u>all</u> other numbers that *cannot* be classified as *rationally-based*.

The ***rationally-based number classification*** should be viewed as a set of *real numbers* which includes all possible *Euclidean* determinations that can be ***geometrically constructed*** from a given, arbitrary length of unity.

It *collates* a disparate assortment of *rational* and quadratic *irrational lengths* together, like $4+(32/62)\sqrt{5}+17\sqrt{35\sqrt{7/1025}}$, whose *individual terms* consist specifically of:

1) *Rational lengths* -- defined as the *quotient* between two *given* integers and portrayed as follows:

$$x_1 = \frac{\Delta}{2a} = \frac{x_1}{1}$$

The *mathematic division* represented above identifies a length x_1 that is determined via ***geometric construction*** performed in accordance with the *Euclidean Mapping Process* specified in *Section 2.3*, where:

- Lengths *Δ* and *2a*, each representing *integer values*, are ***geometrically constructed*** via sole *straightedge* and *compass* using an arbitrary, assigned length of *unity* as a basis
- *Rational length x_1* is identified as the horizontal offset measured from the right side of the rectangle to the point where the diagonal line intersects the horizontal line whose height is unity *(Ref. Figure 2)*

Hence, all *rational numbers* are *Euclidean*! In other words, each and every one can be ***geometrically constructed*** from an arbitrary length which is to be designated as one unit long via only a *straightedge* and *compass*; and

2) *Quadratic irrational lengths* – defined as *all* lengths that can be ***geometrically constructed*** via *Pythagorean Theorem* either from *solely rational lengths* in concert with an infinite variety of mathematical combinations of other *purely rational lengths*, or from their results.

Even after such *rational values* become *transformed* into quadratic *irrational lengths* via *Pythagorean Theorem*, it still remains possible to measure them, as well as to replicate them from a given, arbitrary length of unity.

Mathematically, such ***geometric construction*** process is analogous to calculating respective *root pair values* x_1 *and* x_2 depicted below via *Quadratic Formula* that operates *only* upon sole *rational (or rationally-based) coefficient values a, b,* and *c* that are inherent to, or reside within the specific *Quadratic Equation format* $ax^2+bx+c=0$:

$$x_1; x_2 = [-b \pm \sqrt{b^2-4ac}\,]/2a$$

In conclusion:

- ***Rationally-based numbers*** comprise all *real numbers* which can be ***geometrically constructed*** from a given, arbitrary length of unity
- ***Cubic irrational numbers*** comprise all other *real numbers*; specifically, those which cannot be ***geometrically constructed*** from a given, arbitrary length of unity

Section 9.2 presents various equations which express *mathematical combinations* of *cubic irrational number roots* on their *right-hand sides* that actually can be collated with *rationally-based numerical results* which appear on their respective *left-hand sides*. They consist of:

- A *known*, or *given discrete value* which equals the product of three distinct, but linked, *cubic irrational number roots (Ref. Table 13)*
- A *known*, or *given discrete value* which equals the summation of three distinct, but linked, *cubic irrational number roots (Ref. Table 14)*
- A *known*, or *given discrete value* which equals the summation of paired products of three distinct, but linked, *cubic irrational number roots (Ref. Table 15)*

This unique capability to characterize *cubic irrational roots* in terms of sole *rationally-based coefficients* is reserved only for *Cubic Equation formats.*

Furthermore, *Quadratic Equation formats* do not possess this ability, simply because they require at least one *cubic irrational coefficient* to be present in order to produce a *cubic irrational root pair*.

Section 9.3 asserts that *Cubic Equation formats* pose a complete *demarcation* from their *Linear* and *Quadratic Equation counterparts*.

Such contention prefers an ***extraordinary implication*** upon *Number Theory* by suggesting that equations might assume their very own form in order *to <u>account</u> for the numerical representations included therein*.

This gives rise to a new ***Cubic Equation Uniqueness Theorem*** as described below:

> ***<u>Only</u> Cubic Equations allow solely rationally-based numerical coefficients to co-exist with root sets comprised of cubic irrational numbers.***

This theorem in no way disputes, or contradicts the fact that *cubic irrational root pairs* can, and do exist within *Quadratic Equation formats*.

What is very interesting, predicated upon what was deduced above, is that the only way this can occur is when *coefficients b′* and/or c'' also are *cubic irrational*.

As such, a ***corollary*** to the *Cubic Equation Uniqueness Theorem* appears below:

> ***Cubic irrational root pairs* which appear in *Parabolic Equations* or their *associated functions* require *supporting cubic irrational coefficients*.**

A *logic diagram* is presented in *Section 9.3* for purposes of verifying the above *corollary*.

<u>SECTION 10</u>

Equation Sub-elements first became evident through a novel *missing link transform*, hereinafter referred to as the *Unified Cubic Trigonometric Reduction Equation* (*Ref. Equation 29)*. Such *UCTRE* serves as a *direct conduit* whereby *RST Terminology* embedded within *Cubic Equations* can be *dispensed* into *reduced Quadratic Equations*. Because of such *linkage*, resulting *lower order equations* thereafter house vital *higher order equation* information.

$$\zeta(RST-1)+[(R+S+T)-3RST]\tan\theta+\zeta[(RS+RT+ST)-3RST]\tan^2\theta=0 \quad \textit{[Ref. Equation 29]}$$

SECTION 11

Stemming from *Equation 29*, a set of *supporting fundamental transforms* is determined listed as follows:

- The *SUCTRE* -- see above *(Ref. Equation 30)*
- The *Characteristic Cubic Equation* *(Ref. Equation 31)*
- The *Generalized Cubic Equation* *(Ref. Equation 32)*
- The *Expression for S and T* *(Ref. Equation 33)*
- The *Expression for R and (S+T)* *(Ref. Equation 34)*
- The *Cubic Restitution Equation* *(Ref. Equation 35)*
- The *ζ Relationship to GCE Coefficients* *(Ref. Equation 36)*

Of these, the *Characteristic Cubic Equation (Ref. Equation 31)* contains *coefficients* which are ***inextricably linked*** to the other aforementioned *transforms*. These consist of B, C, and D coefficients comprised of ***RST Sub-element*** combinations as depicted below:

$$B = -(R + S + T)$$
$$C = RS + RT + ST$$
$$D = -RST$$

As such, in addition to serving as *factors* to *cubic roots*, *RST Sub-elements* also permeate, or are embedded deep within the framework, or architecture of constituent *algebraic equation coefficient structures*.

Overall, they perform as *building blocks* that can be *associated* to a patchwork of other aggregate equation assemblages.

SECTION 12

In a sense *Equation 31* may be viewed as a *crossroads* which interconnects a plethora of other *associated transforms* by means of a so-called ***Characteristic Cubic Equation Thruway System***. It embodies various strategically emplaced *Quadratic and Cubic Equation Formats* where *travel* between respective points occurs whenever one format becomes successfully *transformed* into an adjoining one *(Ref. Table 16)*. The process is controlled by a *rigid set of rules (Ref. Table 17)* which accounts for all of the necessary calculations.

Such ***Thruway System*** may be compared favorably to the *software and codes* which led to the development of the ***relational database***, now heavily relied upon in the field of *computer science*. For purposes of introducing *spreadsheets*, it first assumed the form of *System R* in its infancy; but later evolved into SQL, Oracle, and *Excel.*

SECTION 13

The ***coefficient structure*** for any given *Cubic Equation* provides indication of which *methodology* should be employed to *resolve it (Ref. Sections 13.1 thru 13.5)*.

An *interpretation* of such structure consists either of determining its inherent ***R, S, and T sub-element*** *values (Ref. Sections 13.1 and 13.3)*, or discerning whether the coefficients relate to one another in a certain prescribed manner *(Ref. Sections 13.2, 13.4 and 13.5)*.

Section 13.1 applies when an ***R, S, or T sub-element*** *value* equals unity. This causes the *sum* of the *coefficients in* the *Characteristic Cubic Equation (Ref. Equation 31)* to equal zero, as follows:

$$AR^3 + BR^2 + CR + D = 0 \qquad \text{[Ref. Equation 31]}$$
$$A(1)^3 + B(1)^2 + C(1) + D = 0$$
$$A + B + C + D = 0$$

In order to determine whether *Section 13.1* applies to any *given*, or *postulated Generalized Cubic Equation*, it first becomes necessary to transform it into an ***associated Characteristic Cubic Equation*** by applying the respective *lower portion of Table 16 specifications* in accordance with the *Table 17 travel rules*. For example:

$$z^3 + 4z^2 - 15.5z + 7 = 0 \qquad \text{[Given Generalized Cubic Equation]}$$

First, *Equation 36* is applied as follows:

$$\zeta = \tan(3\theta) = \frac{\delta - \beta}{1 - \gamma} \qquad \text{[Ref. lower portion of Table 16]}$$
$$= \frac{7 - 4}{1 - (-15.5)}$$
$$= \frac{3}{16.5}$$
$$= 0.18181818$$
$$3\theta = 190.3048465^o$$
$$\theta = 63.43494882^o$$
$$\tan\theta = \tan 63.43494882^o$$
$$= 2$$

Such that,

$$A = 1$$
$$B = \beta / \tan\theta = 4/2 = 2$$
$$C = \gamma / \tan^2\theta = -15.5/4$$
$$D = \delta / \tan^3\theta = 7/8$$

The resulting *Equation 31* is:

$$AR^3 + BR^2 + CR + D = 0 \qquad \text{[Assoc. Characteristic Cubic Equation]}$$
$$R^3 + 2R^2 - (31/8)R + 7/8 = 0$$

$$A + B + C + D = 1 + 2 - 31/8 + 7/8$$
$$= 3 - 24/8$$
$$= 0$$

Hence, ***Section 13.1 applies*** and:

$$R = 1$$
$$z_R = R \tan\theta = (1)\tan\theta = \tan\theta = 2$$

Values for ***sub-elements S*** and ***T*** are arrived at by using the equation below *(Ref. Section 13.1)*

$$S;T = (1/2)[-(B+1) \pm \sqrt{4D + (B+1)^2}]$$
$$= (1/2)[-(2+1) \pm \sqrt{4(7/8) + (2+1)^2}]$$
$$= (1/2)[-3 \pm \sqrt{7/2 + 9(2/2)}]$$
$$= (1/2)(-3 \pm \sqrt{25/2})$$
$$= (1/4)[-3(2) \pm 5\sqrt{2})]$$
$$= 0.267766953; -3.267766953$$

Then,

$$z_S = S \tan\theta = 0.267766953(2) = 0.535533906$$
$$z_T = T \tan\theta = -3.267766953(2) = -6.535533906$$

Check,

$$z_R^{\ 3} + 4z_R^{\ 2} - 15.5z_R + 7 = 0$$
$$(2)^3 + 4(2)^2 - 15.5(2) + 7 = 0$$
$$8 + 16 - 31 + 7 = 0$$
$$31 - 31 = 0$$
$$0 = 0$$

$$z_S^{\ 3} + 4z_S^{\ 2} - 15.5z_S + 7 = 0$$
$$(0.535533906)^3 + 4(0.535533906)^2 - 15.5(0.535533906) + 7 = 0$$
$$0.153589284 + 1.147186258 - 8.300775543 + 7 = 0$$
$$8.300775543 - 8.300775543 = 0$$
$$0 = 0$$

$$z_T^{\ 3} + 4z_T^{\ 2} - 15.5z_T + 7 = 0$$
$$(-6.535533906)^3 + 4(-6.535533906)^2 - 15.5(-6.535533906) + 7 = 0$$
$$-279.1535893 + 170.8528137 + 101.3007755 + 7 = 0$$
$$-279.1535893 + 279.1535893 = 0$$
$$0 = 0$$

Section 13.3 introduces an *overall*, or *universal* ***cubic resolution capability*** that amplifies or expands upon *fragmented*, or *partially presented prior state-of-the-art techniques (Ref. Section 13.3.6)*.

Hereinafter termed the ***Cubic Resolution Transform (CRT)***, this *newly proposed, overall Cubic Resolution Methodology* serves to unify such *aforementioned theories* into a more powerful, comprehensive algorithm by offering the following unique capabilities:

- It ***directly resolves*** all *Cubic Equations*, *regardless* of what format they may appear in (*Ref. Sections 13.3.3 thru 13.3.5* and *Related Problems 26 thru 33*). In stark contrast, *prevalent present day resolutions* are limited in the sense that they can operate *only* upon *cubic formats* which are *devoid* of their second order terms (*Ref. Section 13.3.6*). Accordingly, they require that *most given Cubic Equations first* become subjected to the additional step of undergoing a *transformation* before resolution can be accomplished (*Ref. Section 13.3.6*)
- It exhibits a *CRT* construction *(Ref. Figure 11)* which now may be applied to such *limited, present day cubic resolutions* in order to allow them to be represented geometrically (*Ref. Section 13.3.6*)
- It readily deciphers whether a given *Cubic Equation* contains *imaginary root sets*. This is accomplished by *manipulating* the known *coefficients* in order to calculate the value of ψ *(Ref. Sections 13.3.1 and 13.3.2, 13.3.4 and 13.3.5)*, such that,

  If $-1 \le \psi \le +1$, *three real roots* exist; if not, an imaginary set applies

  Figure 39 depicts a set of possible *curve scenarios* where, as shown, only the *middle curve* renders *three real roots (Ref. Section 15.3)*
- It affords *three possible format selections* listed as follows, one of whose respective, *coefficient sign conventions* match those specified in any *given*, or *postulated Generalized Cubic Equation* devoid of its second term. Such match-up *must* be conducted *prior* to performing resolution via the ***well-known*** *Trigonometric Solution of the Cubic Equation (Ref. Section 13.3.6)*:
  - $4\cos^3\theta - 3\cos\theta - \cos(3\theta) = 0$ *[Ref. Equation 1]*
  - $4\sin^3\theta - 3\sin\theta + \sin(3\theta) = 0$ *[Ref. Equation 2]*
  - $4\sinh^3 x + 3\sinh x - \sinh(3x) = 0$ *[Ref. Section 13.3.6]*
- It ***mathematically determines*** a *Cubic Equation root* in terms of the following *coefficients* and $\cos(2\varpi)$.

  $z_R = -\frac{1}{3}[\beta - 2\sqrt{\beta^2 - 3\gamma}\cos(2\varpi)]$ *[Derived from Equation 42]*

- It further explains that the property $\psi = \cos(6\varpi)$ is ***coefficient driven*** (i.e.; *fully distinguishable* merely by *coefficient manipulation -- Ref. Section 13.3.3)*, allows for computation of the value of $\cos(2\varpi)$ directly from it, and thereafter enables final algebraic determination of such unknown *Cubic Equation root* z_R. Such approach is consistent with an *inability* to ***trisect*** such 6ϖ angle via *geometric construction*, or ***Euclidean*** *means (Ref. Figure 11).*

Whereas the *roots* to any *given Parabolic Equation* of the form $ax^2 + bx + c = 0$ are *coefficient driven*, it can be resolved solely by *manipulation* of its *coefficients* via the ***well-known*** *Quadratic Formula*) as follows:

$$x_1; x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sub-element Theory has carried on with this tradition in order to surface, or unearth a list of other, new *coefficient driven* properties depicted as follows:

- $\zeta = \tan(3\theta) = \dfrac{\delta - \beta}{1 - \gamma}$ *[Ref. Equation 36]*
- $\psi = \cos(6\varpi) = \dfrac{9\gamma\beta - 2\beta^3 - 27\delta}{2(\beta^2 - 3\gamma)^{\frac{3}{2}}}$ *[Derived from Equation 42]*

More specifically, *coefficient driven* properties can be determined by manipulation of the *Generalized Cubic Equation coefficient structure*. For example, when $\beta = 0$:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$\alpha z^3 + \gamma z + \delta = 0$$

Then,

$$\psi = \cos(6\varpi) = \frac{9\gamma\beta - 2\beta^3 - 27\delta}{2(\beta^2 - 3\gamma)^{\frac{3}{2}}}$$

$$= \frac{9\gamma(0) - 2(0)^3 - 27\delta}{2[(0)^2 - 3\gamma]^{\frac{3}{2}}}$$

$$= \frac{\delta}{2(-\frac{\gamma}{3})^{\frac{3}{2}}} \qquad \textit{[Ref. Equation 41]}$$

<u>In conclusion</u>: From a ***number theory*** *standpoint*, such *previously developed algorithms* now more appropriately should be *categorized* as *sub-classifications* to the *newly proposed*, *universal* ***Cubic Resolution Transform***.

Section 13.2 may be employed to resolve a *Generalized Cubic Equation* of the form shown below when $\beta^2 = 3\alpha\gamma$.

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$\alpha z^3 + \beta z^2 + (\frac{\beta^2}{3\alpha})z + \delta = 0$$

In *Cubic Equations* where this occurs, one of its *roots* is *coefficient driven* and, hence, may be determined as:

$$z = \frac{-\beta + \sqrt[3]{\beta^3 - 27\alpha^2\delta}}{3\alpha}$$

Section 13.4 applies to *Cubic Elliptical Relationships* of the form $b^3 - 2ab^2 + b - c^2 = 0$ *(Ref. Equation 44)*. Such equation types exhibit the *Cubic Equation root* $b_1 = a^2$, and meet the following typical *ellipse properties* afforded in *Figure 12*:

$$a = \overline{AB}$$

$$b = \overline{OA} = a^2 \qquad \textit{[When } \overline{AC} = 1\textit{]}$$

$$c = \overline{AD}$$

Section 13.5 applies to *Generalized Cubic Equations* whose *β* and *γ* terms are equal to *zero*, as follows:

$$\alpha z_R{}^3 + \beta z_R{}^2 + \gamma z_R + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$1z_R{}^3 + 0z_R{}^2 + 0z_R + \delta = 0$$

$$z_R{}^3 + \delta = 0$$

As demonstrated in *Section 13.5*:

$$z_R{}^3 + \zeta = 0$$

$$z_R{}^3 = -\zeta = (R\tan\theta)^3$$

$$z_R = \sqrt[3]{-\zeta} = R\tan\theta \qquad \textit{[Ref. Equation 46]}$$

The *equation form* $z_R{}^3 + \zeta = 0$ results from *GCE's* when $\gamma = -\beta z_R$, as:

$$\alpha z_R{}^3 + \beta z_R{}^2 + \gamma z_R + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$1z_R{}^3 + \beta z_R{}^2 - (\beta z_R)z_R + \delta = 0$$

$$z_R{}^3 + \delta = 0$$

SECTION 14

The *practice* of performing *mathematical operations* upon equation *coefficients* is neither new nor unique to number theory.

The *Quadratic Formula*, depicted below, perhaps stands as its most *famous* and *significant* exponent by expressing *root set pairs*, designated below as x_1 and x_2, as little more than ***mathematical manipulations*** of only *intrinsic coefficients a*, *b*, and *c* **harbored** within *Parabolic Equations* of the form $ax^2 + bx + c = 0$:

$$x_1; x_2 = [-b \pm \sqrt{b^2 - 4ac}]/2a$$

The ***Characteristic Cubic Equation Thruway*** *System* enhances upon this *practice* by enabling *mathematical operations* to be performed upon associated *equation formats* through a *conversion process*, or *transformation* which internally ***links*** resident *coefficient structures (Ref. Table 16)*.

Curve Mapping instead *mathematically operates* upon just one particular *coefficient* structure, or equation *format* at a time *(Ref. Section 14)*. It determines *sets*, or *families* of *coefficient permutations* comprised of intrinsic ***RST terminology***. Hence, a gateway for ***Equation Sub-element*** *categorizations* becomes realized.

Equation Sub-element Curve Mapping Theory maintains that a *stationary parabolic or cubic curve shape* exhibits a singular ***equation format structure*** but, nevertheless, may be characterized by a *multiplicity* of *intrinsic mathematical expressions*, all of which identify *relative position* away from a ***pre-selected point*** *in space (Ref. Section 14)*.

Such concept is further characterized by introducing a *relativistic approach* which applies a *mobile origin* that is *perceived* to move about to *pre-selected points* upon an orthogonal *grid pattern*, thereby affording *different perspectives* with respect to such *stationary point*.

Now, *Parabolic* and *Generalized Cubic Function* ***coefficient structures*** are considered to be the very best possible candidates to represent respective *Quadratic* and *Cubic Function format* ***classifications*** because:

1) They limit *higher order expressions* to just one variable (or unknown), thereby promoting a *simplified mathematical analysis*, and
2) They allow for the greatest amount of *mathematical flexibility*.

However, *circles*, *ellipses*, *hyperbolas*, *Complex Quadratic Functions* and *Complex Cubic Functions* such as the one designated below also may qualify for subsequent treatment:

$$\alpha z^3+\beta z^2+\gamma z+\delta=y^2$$

Accordingly, ***selected*** *Parabolic* and *Generalized Cubic Function coefficient structures* are listed below:

$ax^2+bx+c=y$ *(Ref. Section 14.1)*

$\alpha z^3+\beta z^2+\gamma z+\delta=y$ *(Ref. Section 14.2)*

The prospect of *realizing location* from a *singular point in space* is comparable to *pinpointing* an object by sonar, or wave reflection, whereby its *distance* away is easily calculated by assessing the *time* it takes for the wave to propagate to the object, multiplied by a predetermined *velocity* as it travels through a *known* medium.

For this study: *Triangulation*, which enables a position to be trigonometrically determined with respect to *two fixed points*, applies only when such second identified point is used to attribute an orientation for a *Cartesian Coordinate System* intended for use in a *Curve Mapping* analysis.

Various *travel route scenario* examples for the *Parabolic Curve mapping process* are listed below:

a) Those which occur across its *root sets (Ref. Section 24 Related Problem Nos. 35 and 36)*;
b) Those which occur directly along a *Parabolic Curve (Ref. Section 24 Related Problem No. 38)*; and
c) Those which occur along any other *selected route*, such as over a *circular path* between *root sets (Ref. Section 24 Related Problem No. 40)*.

Parabolic *and* ***Generalized Cubic Curve Mapping methodology*** consists of:

1) A *Singularity Proof* stating that all *family curves superimpose* onto a *parent curve* of *identical shape (Ref. Sections 14.1.1, and 14.2.1)*;
2) An accompanying *Algorithm* which reveals that a singular, stationary curve in space may be referred to by a *multiplicity* of *independent mathematical functions* which afford tracking or mapping capabilities *(Ref. Sections 14.1.2, and 14.2.2)*; and
3) An *Application subsection* which demonstrates precepts developed earlier by focusing upon certain detailed relationships that exist between *families of identically shaped curves (Ref. Sections 14.1.3, and 14.2.3)*.

With particular regard to ***Parabolic Curve Mapping***:

- A ***Singularity Proof*** *(Ref. Section 14.1.1)*, comprised of a *threefold mathematical analysis*, validates an ability to *superimpose* differing *Parabolic Curve Functions* of the *same exact curve shape* onto one another. *Functions* presented below, demonstrated to meet such criteria, maintain their own *independent* origins located respectively at Points *O*, *A*, and *B* *(Ref. Figure 13)*:
    - $ax_O^2 = y_O$
        - $ax_A^2 + y_M = y_A$ *(Where* $x_A = x_O$*)*

          Such that,

          $y_O + y_M = y_A$

          [Whereas the term y_M can assume an *infinite* number of values, its associated term y_A becomes *compensating* because it *adjusts* for values of y_O which satisfy the function:

          $y_O = ax_O^2$]
            - $ax_B^2 + bx_B + c = y_B$ *(Where* $x_B = x_A + x_M$ and $y_B = y_A$*)*

              Such that,

              $b = -2ax_M$

              $c = y_M + \frac{b^2}{4a}$

              [Likewise, as another associated term of y_M, *c* becomes *compensating* because it *adjusts* for values of $b^2/(4a)$ which satisfy the function $ax^2 + bx + c = y$]

  Such *Singularity Proof* furthermore is validated *pictorially* where the above functions are all plotted with respect to a *common origin* located at *point A*, without compensating for the relative origin assignments which were originally applied *(Ref. Figure 14)*. Nevertheless, they still maintain their original *identical curve shapes* in the form of step functions with respect to the curve whose low point passes through *point A*. The identically shaped curves appear *displaced* from one another by lengths equal to these respective step functions, or distances initially selected between respective origins *(Ref. Figure 13)*. So, once compensation for origin difference becomes accounted for, via translation, the three curves become one, or coincide, just as is forecasted in this *Singularity Proof*.

  In conclusion, a *given Parabolic Function* along with *two transforms* derived from it, as listed above, produce a total of three *identically shaped Parabolic Curves* ***all*** of which *occupy the same exact coordinates*, where each merits its *own* independent *perspective*, or *point of origin*.

  Accordingly, it now becomes possible to *associate* given *Parabolic Curves* with a plethora of other *Parabolic Functions* which exhibit their same exact shape.

- An associated ***Algorithm*** *(Ref. Section 14.1.2)* interprets *Figure 13* as a grid of *potential* ***origin relocations*** away from the *low point* (or *high point*) of any *given Parabolic Curve*.

  Such *given curve* may be viewed, or perceived from *different elevations* or perspectives, each offering harbor for a multiplicity of *Point B origin placements (Ref. Figure 13)*.

  The *algorithm* is premised upon a set of ***properties***, or *curve attributes*, which identify *principal characteristics* which govern the very shape of any *given Parabolic Curve*.

  With respect to such *given stationary Parabolic Curve*, for any arbitrarily selected *origin assignment*, such *properties* relate to the coefficients of the *General Parabolic Equation* $\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}=0$ and, therefore, also to its respective *root sets* through the *universally known Quadratic Formula*:

  $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

  In particular, for any specified elevation, or x-axis assignment, *root sets* vary depending upon which particular *origin relocation* is selected. In other words, for any specific elevation, an unlimited number of *root sets* apply, all of which delineate respective distances from selected origins to intersection points on the *given Parabolic Curve*.

  This is easily recognized by observing that measurements from *Point B* to respective x_1 and x_2 endpoints *change* as such point relocates along the x-axis *(Ref. Figure 13)*.

  Hence, x_1 and x_2 *root sets*, as generated from a relocated origin, can be of any combination of *desired proportions*.

  Three example *proportional sets*, as specified below, are further examined (Ref. *six bullets on page 145)*:

  - $x_1=\tan\theta$
  - $x_2=-1/\tan(2\theta)$
    - $x_1=\tan\theta$
    - $x_2=-1/\tan\theta$
      - $x_1=\tan\theta$
      - $x_2=-\tan(2\theta)$

  A *series* of *Parabolic Curves* each of which bear the *exact curve shape* and *same low point value* as the following *Parabolic Function*, but whose respective *generated root sets* along the x-axis exhibit the above stated *three root structures* are determined *(Ref. Section 24 Related Problem Number 39)*

  $$3x^2-200=y$$

  Accordingly, the *algorithm* serves to connect *families* of *Parabolic Function* coefficient relationships together via their association to tan θ.

- An ***Application subsection*** *(Ref. Section 14.1.3)* relates *essential algorithmic relationships* for the *specific condition* reiterated below *(Ref. Table 18)*:
    - $x_1 = \tan\theta$
    - $x_2 = -1/\tan(2\theta)$

  This involves determining the following relationship *over a range of angles*, and their *associated tangents*, by applying respective coefficient values calculated from equations presented in *Section 14.1.2*:

  $$x_M = -\frac{b}{2a}$$

  Respective *Parabolic Curve* low points, y_M then are calculated as indicated.

  Figure 15 portrays *eighteen identically shaped curves*, all belonging to the *Parabolic Curve Family*:

  $$\mathrm{ax}^2 = \mathrm{y}$$

  Table 19 is a *Figure 15* tabulation, wherein examples of how such values were arrived at are provided in the write-up.

  Figure 16 demonstrates that *all eighteen curves*, by virtue of the fact that they're identically shaped, entirely overlap one another.

  This is proven via the following ***Normalization Transformation for Parabolic Functions***:

  $$a(x + x_M)^2 + b(x + x_M) + (c - y_M) = y$$

  For *all eighteen curves*, *transformation results* indicate the exact same value for y for any value of x, thereby verifying *singularity* of the *curve family (Ref. Table 20)*.

  A later *application* determines a *Parabolic Curve Function* which exhibits specific *low point coordinates*, while passing through the *right-hand root* of the function $3x^2 - 200 = y$ and bearing its identical curve shape.

  A second *Parabolic Curve Function* of similar credential, expect for the fact that it can possess different *low point coordinates*, is illustrated in order to emphasize the preponderance of other available *family curves (Ref. Figure 17)*.

Thereafter, *Figure 18* reconciles the four identically shaped Parabolic Curves by demonstrating that they coincide after applying the *Normalization Transformation for Parabolic Functions.*

With particular regard to ***Generalized Cubic Curve Mapping***:

- A ***Singularity Proof*** *(Ref. Section 14.2.1)*, validates that two equations, as denoted below, generate *identically shaped cubic curves*:

  1) $z^3+\beta' z^2+\gamma' z+\delta' = y_{TRANSFORMED}$

  2) $z'^3+\sigma z'^2+\nu = y'$

  Where,

  $$\sigma = -\sqrt{\beta'^2-3\gamma'}$$

  $$\nu = \frac{1}{27}[2\beta'^3-(2\beta'^2-6\gamma')\sigma-9\beta'\gamma'+27\delta']$$

  The *second curve* is established by determining z-axis locations where the slope of the *first curve* is equal to zero. This is achieved by taking the derivative of such *upper curve* and setting it equal to zero, which produces:

  $$z_A, z_B = \frac{1}{3}[-\beta' \pm \sqrt{\beta'^2-3\gamma'}]$$

  The *second function* and associated values for its *coefficients* are determined by specifying the respective *root values* for the first *function* with respect to a *relocated origin* which is displaced a value z_B away from the *initial origin*, as follows:

  $z_R' = z_R - z_B$

  $z_S' = z_S - z_B$

  $z_T' = z_T - z_B$

  Where,

  $$(z'-z_R')(z'-z_S')(z'-z_T') = y'$$

  $$(z'-z_R')[z'^2-(z_S'+z_T')z'+z_S'z_T'] = y'$$

  $$z'^3-(z_R'+z_S'+z_T')z'^2+(z_R'z_S'+z_R'z_T'+z_S'z_T')z'-z_R'z_S'z_T' = y'$$

  $$z'^3+\sigma z'^2+\tau z'+\nu = y'$$

  Such repositioning <u>always</u> determines $\tau = 0$ *(Ref. write-up)* for the following overall function.

  $z'^3+\sigma z'^2+\tau z'+\nu = y'$

  Accordingly, a *Generalized Cubic Function* of the form $z'^3+\sigma z'^2+\nu = y'$ always exhibits an *origin* which is *vertically aligned* with a point upon such curve whose slope is equal to zero.

  Another format possibility for the function $z'^3+\sigma z'^2+\tau z'+\nu = y'$ occurs when $\sigma = 0$ as follows:

  $z'^3+\tau z'+\nu = y'$

  This characterizes a function whose associated *straight line* and *Perfect Cubic Function (Ref. Section 24 Related Problem Number 42) intersect* at *points* which are located upon the respective *vertical projections* of its *three roots*.

- An associated ***Algorithm*** *(Ref. Section 14.2.2)* theorizes that a *Parent Generalized Cubic Function* exists which can fully characterize any *given Generalized Cubic Curve* in every respect.

  Moreover, it contends that its *coefficient structure* can be determined by *mathematically interpreting* the *values* of certain ***properties*** exhibited by such *given Generalized Cubic Curve*.

  Figure 20 portrays these two aforementioned *family curves* in consonance with their identically shaped *(Ref. Table 22)* associated *Parent Generalized Cubic Function*, the three of which are itemized below:

  1) $z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$
  2) $z'^3 + \sigma z'^2 + \nu = y'$
  3) $z'^3 + \sigma z'^2 = y''$

  In conclusion, *Generalized Cubic Curves* possessing *identical shapes* may be superimposed onto a single *Cartesian coordinate system*, placed at various strategic locations which are *traceable* to various pre-determined, mutually independent *Generalized Cubic Functions*.

  Such activity enables sets, or *families* of *Generalized Cubic Functions* to become linked, and/or charted into *desirable arrays*, which in turn may be classified with respect to their very *root structures*.

  Whereby such *root structures* specify actual respective horizontal spans between locations where *Cubic Functions* cross the x-axis, they now may be categorized in terms of so-called ***RST Spreads*** *(Ref. section 15 below)*.

- An ***Application subsection*** *(Ref. Section 14.2.3)* depicts *variability in curve shape as realized when* $\xi = \tan(3\theta)$ changes value with respect to the following *Cubic Parent Curve (Ref. Figure 21 and Table 23)*:

  $$z'^3 - 3\sqrt{1+\zeta^2}\, z'^2 = y''$$

  Figure 22 shows *Equation 25* with respect to its *Parent Cubic Function, expressed above for the particular case when* $\zeta = \sqrt{3}$, thereby indicating identical curve shape.

In conclusion, an *equation* for a *fixed curve in space* is not absolute, but instead becomes altered depending upon an *observer's perspective*. Viewers who perceive the fixed curve from different vantage points stipulate alternate equations which also precisely depict it.

SECTION 15

RST Spreads represent an amalgamation of *root set spacings* that apply to any *given Generalized Cubic Function (GCF)*. They accrue as the z-axis becomes displaced *vertically* with respect to such curve, now considered to be *stationary*.

Hence, they depict an assortment of relative *root set* spans which exist along such *GCF* as it becomes viewed horizontally from *different elevations*.

More specifically, *RST Spreads* may be distinguished as *deviation* from a ***three dimensional space norm*** where, for purposes of this treatise:

- Such ***norm***, or ***benchmark*** hereinafter referred to as the *3θ Cubic Tangent Function* (or just the *3θ Cubic Function*) is to be represented as the *function* for *Equation 22* as follows, selected because *RST Spreads* become *useful* when they are categorized, or assembled, with respect to *3θ Cubic Functions* which they modify, or belong to:

$$z^3 - 3\zeta z^2 - 3z + \zeta = y \qquad \textit{[Ref. Equation 22]}$$

Where,

$$z_R = \tan\theta_R = R\tan\theta = \tan\theta$$
$$z_S = \tan\theta_S = S\tan\theta = \tan(\theta + 120^o)$$
$$z_T = \tan\theta_T = T\tan\theta = \tan(\theta + 240^o)$$
$$\theta_R + \theta_S + \theta_T = 3\theta + 360^o = 3\theta$$

- A ***three dimensional space norm*** is to be represented via the ***volumetric expletive RST***, otherwise expressed as the negative of *coefficient 'D' of the Characteristic Cubic Equation as follows*:

$$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$
$$AS^3 + BS^2 + CS + D = 0$$
$$AT^3 + BT^2 + CT + D = 0$$

Where,

$$A = 1$$
$$B = -(R + S + T)$$
$$C = RS + RT + ST$$
$$D = -RST$$

Such *benchmark curve* can be associated to *three* other curves of *identical shape* as follows *(Ref. Section 15.1)*:

1) The *Family Cubic Function* which is to be represented by the *Generalized Cubic Function* $z^3 + \beta z^2 + \chi z + \delta = y$ when $\alpha = 1$;
2) Its associated *Intermediate Cubic Function* which is to be represented by the *Function* $z'^3 + \sigma z'^2 + \nu = y'$ *(Ref. Generalized Cubic Curve Mapping Singularity Proof write-up above)*; and
3) Its *Parent Cubic Function* which is to be represented as $z'^3 + \sigma z'^2 = y''$. Notice that the *Parent Cubic Function* is *identical* to the *Intermediate Cubic Function* above because:

$$z'^3 + \sigma z'^2 + \nu = y'$$
$$z'^3 + \sigma z'^2 = y' - \nu$$
$$z'^3 + \sigma z'^2 = y''$$

Whereby its first *root*, ascertained when setting the *function* equal to zero, may be determined via reduced *Linear Equation* as follows:

Where,

$$z'^3 + \sigma z'^2 = 0$$

$$z' + \sigma = 0$$

$$z' = -\sigma$$

Section 15.1 examines these four *Cubic Functions* and demonstrates how they are associated for the *specific* ***RST Spread*** of:

$R = 1$

$S = 4$

$T = 1/2$

In particular, for these *Cubic Functions*:

- *Table 24* enumerates determined θ and 3θ angles, associated tangent information, *coefficient* and *root* calculations
- *Table 25* validates that associated Δ and ε property values are identical, thereby assuring that curves all are of exactly the same shape; it also renders z_A and z_B location details which express how curve shapes are shifted with respect to one another
- *Figure 23* indicates that such sought after ***RST Terminology*** occurs at an *ordinate location* of $y = 53.54992896$ upon the *3θ Cubic Function*, validated as follows:
  - j_1/j_2 = 1.118033989/2.236067978= ½ to 1 = T
  - j_2/j_2 = 2.236067978/2.236067978= 1 to 1 = R
  - j_3/j_2 = 8.944271911/2.236067978= 4 to 1 = S
- *Table 26* presents their respective plots
- *Figure 24 illustrates the relative positioning of the Cubic Family Curve* with respect to its *associated Intermediate and Family Cubic Curves*
- *Figure 25* characterizes the *relative positioning of the Cubic Family Curve* with respect to its *associated 3θ Cubic Curve*

Next, the above *curve set* is *linked* to the *Characteristic Cubic Equation (Ref. Equation 31)*, whose calculated coefficients were tabulated back in *Table 24*. Via comparison between respective Δ and ε property values, it then is determined that the respective curve shapes are not identical.

Accordingly, a *second set* of *associated curves* is then developed in similar fashion to that of the first, where instead the *Characteristic Cubic Function* is applied as the *Family Cubic Function*. Thereafter, *Figure 26, Figure 27, Figure 28, Table 27,* and *Table 28* are produced using the very same approach described above.

Now, the fourth term of the *Characteristic Cubic Equation (Ref. Equation 31)* characterizes ***RST Terminology*** as a *volumetric expletive (Ref. Section 15)*. This is evidenced below:

$$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$

Where,

$$D = -RST$$

Therefore, as *building blocks to sets of established 3θ Cubic Functions*, ***RST Spreads*** specify *the very realms of three dimensional space which these equations* occupy *(Ref. Section 15.2)*.

RST Spreads for the *norm* are constructed by reconstituting the *3θ Cubic Function* into *equation form*, and then solving for its roots as follows:

Where,

$$z^3 - 3\zeta z^2 - 3z + \zeta = y$$

$$z^3 - 3\zeta z^2 - 3z + (\zeta - y) = 0$$

$$(z - z_f)(z^2 + Mz + N) = 0$$

$$z^3 + (M - z_f)z^2 + (N - Mz_f)z - Nz_f = 0$$

Comparison of coefficients yields,

$$M - z_f = -3\zeta$$

$$M = -3\zeta + z_f$$

$$-Nz_f = (\zeta - y)$$

$$N = -\frac{(\zeta - y)}{z_f}$$

Such That

$$z_1, z_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-M \pm \sqrt{M^2 - 4N}}{2}$$

$$= \frac{1}{2}[3\zeta - z_f \pm \sqrt{(z_f - 3\zeta)^2 + 4\frac{(\zeta - y)}{z_f}}] \sqrt{3}$$

Figure 29 depicts an associated ***RST Spread*** for the *norm* when $\zeta = \tan(3\theta) = \tan 60^\circ = \sqrt{3}$.

Therein, a *real root region* is bounded below by the y_A horizontal offset as it extends to the left until it intersects the point of non-zero slope on the *3θ Cubic Curve*; and is bounded above by the y_B horizontal offset as it extends to the right until it intersects another point of non-zero slope on the *3θ Cubic Curve.*

With respect to *Figure 29*, the *three dimensional space* becomes expressed as the *product* of R, S, and T for any given value of z within the *real root region*. As such, viewing *Figure 29* along the chart's abscissa renders a volume which equates to the product between S and T for each horizontal offset examined, as R equals unity.

Figure 30 enhances upon *Figure 29* by showing vertical lines drawn through respective z_R, z_S, and z_T roots of the *norm*. It is observed that R*, S, and T Curves*, when crossing such vertical lines, continuously do so at the same elevations, thereby ***demonstrating interchangeability***. Such affinity also may be attributed to *other elevations* upon the *norm*.

Table 29 gives the associated plot for *Figure 30* (as well as *Figure 29*)

Elevation value interchangeability, as described above, is considered to be an ***RST Spread attribute***, or *feature* which serves to *identify* an underlying intrinsic quality which otherwise remains hidden within *Cubic Functions*. *Attributes* become further linked to *properties*, or innate *capabilities* of *Cubic Functions* and their associated *formats*.

Additional *attributes* are represented as *coefficients* of the *3θ Cubic Function*. This is further disclosed in *Figure 31* which depicts *two straight lines* and a *new Cubic Function* of exact shape to the *norm*, with the only exception being that it *rides* below it by a distance of ζ

The *two straight line depictions* represent respective *second* and *third term coefficients* of the *norm*, designated using symbols evidenced in the *Function* of the *Generalized Cubic Equation (Ref. Equation 32)*, and relegated to the unknowns z_f, z_1, and z_2 established above, where:

$$\beta = -(z_f + z_1 + z_2) = y_\beta$$

$$\gamma = -(z_f z_1 + z_f z_2 + z_1 z_2) = y_\gamma$$

The *new Cubic Function* of exact shape to the *norm* is of the following form:

$$z_f z_1 z_2 = -\delta' = y_{\delta'}$$

Such that,

Its ordinates represent associated ***volumetric depictions in linear fashion***, another *attribute*, which truly correspond to the product of any of the z_f, z_1, *and* z_2 ***Spreads*** which reside in the *real root region* on the *3θ Cubic Curve,* regardless of elevation*.*

Figure 32 represents sixteen *3θ Cubic Functions* which exhibit various *arbitrarily selected* 3θ values. Such mapping reports the *variability* evidenced by the *3θ Cubic Function* as it undergoes change in its *fundamental property* ζ.

Figure 33 portrays associated R Values for the various *3θ Cubic Curves* presented in *Figure 32.* For any ordinate selected, representing a *constant value* for R, *Figure 33* illustrates just how much *shape change* occurs to *Figure 32 3θ Cubic Functions* while moving to the right, or increasing in z value; where*, Cubic Curve shape* itself may be viewed as another ultimate property.

Another *attribute* is the *z-axis threshold* at which *S and T values*, respectively, *start* becoming imaginary.

In *Figure 33* the z-axis depicts a range between $-\sqrt{3}$ thru $+\sqrt{3}$ representing thresholds below and above which *S and T values*, respectively, *start* becoming imaginary *(Ref. Table 30)*. This is in stark contrast to *R values*, as plotted on the y-axis, which remain real from negative infinity thru positive infinity.

Within their respective *real root regions*, *S and T Curves* associated with the *3θ Cubic Curve Sets* expressed in *Figure 32* are depicted in *Figure 34* where,

S represents the Lower portion, and T pertains to the upper portion of each curve. *S and T Curves* are joined, or connected, at respective left-most and right-most portions of each curve, respectively. Accordingly, *real root regions* are different for each *S and T Curve* represented *(Ref. Figure 35, Figure 36,* and *Figure 37)*. *Table 31* represents the basis for such plot by charting *RST Curves* with respect to 'z'. For each *3θ Cubic Curve*, it indicates the spans over which the *S and T Curves* remain real and locates exactly where they become imaginary. Therein, respective R, S, and T values are determined as follows:

$$R = \frac{z_f}{\tan\theta'}$$

$$S = \frac{z_1}{\tan\theta'}$$

$$T = \frac{z_2}{\tan\theta'}$$

As a final illustrative example, an ***RST Spread*** is developed for the associated *function* of *the 3θ Cosine Cubic Equation* given below (*ref. Section 15.3 and Equation 1)*. This above nomenclature adds to that of the *3θ Cubic Function*, which really denotes a short-hand notation for the *3θ Tangent Cubic Function*. *Figure 40,* developed through calculations expressed in *Table 32*, portrays an ***RST Spread***, superimposed over the *3θ Cubic Function*, whose S and T Terminology remains real only within the range $-1 \le \cos\theta = z = z_f \le +1$.

$$z^3 - 0z^2 - (3/4z) - \tau/4 = y$$

Section 15.4 depicts an *associated set* of four *identically shaped curves (Ref. Figure 41)*. The *last three* depicted below were derived from the first, which *typically* depicts virtually any *given, specific Generalized Cubic Function*:

$$z^3+\beta' z^2+\gamma' z+\delta' = y_{TRANSFORMED}$$ *[Ref. Section 14.2]*

$$z^3+1.4z^2-3.2z-0.84 = y_{TRANSFORMED}$$ *[Given Generalized Cubic Curve]*

$$z'^3+\sigma z'^2+\nu = y'$$ *[Ref. Section 14.2.1]*

$$z'^3-3.4z'^2+3.768 = y'$$ *[Associated Displaced Family Curve]*

$$z''^3+\beta_{3\theta} z''^2+\gamma_{3\theta} z''+\delta_{3\theta} = y''$$

$$z''^3-3\zeta z''^2-3z''+\zeta = y''$$

$$z''^3-1.6z''^2-3z''+\frac{1.6}{3} = y''$$ *[Associated 3θ Cubic Curve]*

$$z'''^3+\sigma_{3\theta} z'''^2+\nu_{3\theta} = y'''$$

$$z'''^3-3.4z'''^2+1.54133333 = y'''$$ *[Associated 3θ Displaced Family Curve]*

The above *example* applies portions of a *Section 15.4 derivation* which indicates that the *root structure* for any *given Generalized Cubic Function* can be characterized, or reduplicated, by an associated ***RST Spread*** contained within a *3θ Cubic Function* of the same exact *curve shape.* Hence ***RST Spreads***, inherent within ***3θ Cubic Functions*** characterize the *root structures* for <u>all</u> *Generalized Cubic Functions*.

RST Spreads may qualify either as exact *roots* to certain *equations*, or as multiples thereof. Examples follow:

The respective roots for any given *Characteristic Cubic Equation (Ref. Equation 31)* are designated as actual ***R, S, and T Equation Sub-elements*** themselves *(Ref. Section 11.2)*. This is easily evidenced by virtue of the fact that:

$$(q-R)(q-S)(q-T)=0$$

In contrast, ***R, S, and T Equation Sub-elements*** also represent *respective factors*, as indicated above, to all *Generalized Cubic Equation* roots z_R, z_S, and z_T *(Ref. Section 11.3)*.

SECTION 16

Various functions are addressed which exhibit the exact same curve shape as the ***Perfect Cube Parent Function*** *presented below*, and hence belong to its family:

$$y = z^3$$ *[Ref. Equation 47]*

Curves represented by the <u>format</u> expressed below fall within this *family, or set* of *Cubic Curves* , evidenced by the fact that they all exhibit a *singular curve shape* which matches that of the *Perfect Cube Parent Function*, no matter what value of 'a' is applied *(Ref. Section 16.1)*:

$$y = (z \pm a)^3$$

Such *family function* $y=(z\pm a)^3$ is deemed the ***Fundamental Symmetric Cubic Equation*** when $a=1$ *(Ref. Section 13.2)*.

In ***Section 16.2***:

Setting $\gamma'=\beta'^2/3$ when $\beta'=\pm 3a$ below gives:

$$y_{TRANSFORMED}=z^3+\beta' z^2+\gamma' z+\delta' \qquad \text{[Ref. Section 14.2]}$$
$$=z^3+\beta' z^2+\frac{\beta'^2}{3}z+\delta'$$
$$=z^3\pm(3a)z^2+\frac{(\pm 3a)^2}{3}z+\delta'$$
$$=z^3\pm(3a)z^2+\frac{9a^2}{3}z\pm(a^3-a^3)+\delta'$$
$$=(z^3\pm 3az^2+3a^2z\pm a^3)\mp a^3+\delta'$$
$$=(z\pm a)^3\mp a^3+\delta'$$
$$y_{TRANSFORMED}\pm a^3-\delta'=(z\pm a)^3$$
$$\sqrt[3]{y_{TRANSFORMED}\pm a^3-\delta'}=z\pm a$$
$$\sqrt[3]{y_{TRANSFORMED}\pm a^3-\delta'}\mp a=z$$

Since the *Cubic Resolution approach* presented in *Section 13.2* also applies for the specific case when $\gamma'=\beta'^2/3$, it ***is analogous to*** the solution afforded above when $y_{TRANSFORMED}=0$ as follows:

$$z=\sqrt[3]{y_{TRANSFORMED}\pm a^3-\delta'}\mp a$$
$$=\sqrt[3]{0\pm(\beta/3)^3-\delta'}\mp(\beta/3)$$
$$=1/3(-\beta+\sqrt[3]{\beta^3-27\delta'})$$

Section 16.3 portrays the following three *family curves (Ref. Figure 42)*:

$y=z^3$ *[Ref. Equation 47]*

$y=(z\pm a)^3$ *[Ref. first family curve]*
$=(z+7)^3$ *[That is, $+a=+7$ or $-a=-7$]*

$y=(z\pm a)^3+13.257$ *[Ref. second family curve]*
$=(z+7)^3+13.257$

Therein, a *relativistic interpretation* is applied to verify that all *three curves* are identically shaped.

Section 16.4 presents two applications demonstrating:

- When and how the equation $\sqrt[3]{y_{TRANSFORMED}\pm a^3-\delta'}\mp a=z$ can be used
- How *specific values* contained within the *table* expressed within *Figure 42* can be obtained by applying a *relativistic interpretation*

SECTION 17

Equation 48 represents a **new** significant *linear* relationship between tan θ and its associated $\zeta = \tan(3\theta)$ function.

$\tan\theta = -(\frac{J}{F})\zeta$ *[Ref. Equation 48]*

Section 17.1 gives its derivation such that *factors F and J*, shown below, represent *manipulations* of *Characteristic Cubic Equation 31 coefficients*:

$$F = 2[3D - B]$$

$$J = 3(B + C) - (D + 1) \pm G$$

Where,

$$G = \pm\sqrt{9(B^2 + C^2) + D^2 + 14BC - 6BD + 6CD + 1 + 6B - 6C - 34D}$$

Section 17.2 determines yet another *special case circumstance* of the *Generalized Cubic Equation*, hereinafter to be known as the *J-Function Cubic Expression (Ref. Equation 49)* as follows:

$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$ *[Ref. Equation 32]*

$J^3 + (3F)J^2 - 3(\frac{F}{\xi})^2 J - F(\frac{F}{\xi})^2 = 0$ *[Ref. Equation 49]*

Such that when $\alpha = 1$ and $J = z$, by equating like terms:

- $F = \frac{\beta}{3}$
- $F\sqrt{\frac{-3}{\gamma}} = \xi$
- $-F(\frac{F}{\xi})^2 = \delta$

Section 17.2 concludes by presenting brief examples of how the *J-Function Cubic Expression* may be applied.

Section 17.3 shows that the *J-Function Cubic Expression* equates to the *3θ Cubic Function* as follows:

$J^3 + (3F)J^2 - 3(\frac{F}{\xi})^2 J - F(\frac{F}{\xi})^2 = 0$ *[Ref. Equation 49]*

Multiplying thru by $-(\frac{\zeta}{F})^3$ and substituting for *Equation 48 renders:*

$$-(\frac{\zeta}{F})^3 J^3 - 3\zeta(\frac{\zeta}{F})^2 J^2 + 3(\frac{\zeta}{F})J + \zeta = 0$$

$$-(\frac{\zeta}{F})^3[(-F/\zeta)\tan\theta]^3 - 3\zeta(\frac{\zeta}{F})^2[(-F/\zeta)\tan\theta]^2 + 3(\frac{\zeta}{F})[(-F/\zeta)\tan\theta] + \zeta = 0$$

$$\tan^3\theta - 3\zeta\tan^2\theta - 3\tan\theta + \zeta = 0$$

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

Section 17.4 presents two *methods* in which *Equation 49* can be used to determine respective values for $\zeta = \tan(3\theta)$ and tan θ given their *desired ratio*.

SECTION 18

Along with *Equation 36*, other *equation formats* now may be *specified* which link *trigonometric values* of an angle to those of one-third its size.

Inherent *coefficient structures* provide a pathway for *geometric construction* which associates such two trigonometric entities.

This is not the same thing as performing a *Euclidean trisection* because *certain independent information* which is contained within *such coefficient structures also* needs to be assessed, *in addition to* that which is *directly associated* with *a given angle 3θ*.

Three equation type *categories* are afforded below which encompass variations in *coefficient structures*:

1) Those comprised solely of *rationally-based coefficients (Ref. Section 9.1)*;
2) Those comprised solely of *cubic irrational coefficients*, or those which are not *rationally-based (Ref. Section 9.1)*; and
3) Those which contain a *combination* of *coefficients* fitting *Category 1* and *Category 2* descriptions.

A brief list of ***salient equation formats*** which can be *portrayed* and, thereby *further characterized* by *such* ***geometric construction*** is as follows:

- *An Equation 1 Reduction (Ref. Equation 4)*
- The *SUCTRE – A Quadratic Equation (Ref. Equation 30)*
- *The Tan θ to ζ Linearity Expression (Ref. Equation 48)*
- Equations resulting when $z_R = -1/\tan(3\theta) = -1/\zeta$
- *Complex Quadratic Equations for the Angle Trisector Triangle (Ref. Equation 50)*
- Equations emulated by the *Cosine Circle*

Section 18.1 indicates that *Equation 4* qualifies either as a *Category 2* or *Category 3 equation type format.*

$$\cos^2\theta + (\frac{2\tau\lambda - 5}{6\lambda})\cos\theta - \frac{\tau}{2\lambda} = 0$$ *[Ref. Equation 4]*

Its *coefficient values* can become calculated once $\lambda = \sin(3\phi)$ is determined from a given value of $\tau = \cos(3\theta)$ via the relationship:

$$\sin\phi = \frac{1}{2\cos\theta}$$

The $\cos\theta$ can be *constructed geometrically* via the *Euclidean Mapping Process* defined in *Section 2.3*.

As such, *Equation 1* can *be reduced further*. Although *present day conjecture* is that such equation is ***irreducible***, reduction becomes precipitated simply by supplying applicable *irrational coefficients*, as determined by mathematical calculation.

Section 18.2 demonstrates that the *SUCTRE* can exist either as a *Category 1*, *Category 2*, or *Category 3 equation type*.

Its coefficients may be expressed in terms of ζ and arrangements of *coefficients* contained in the *Characteristic Cubic Equation*, as indicated below:

$$a = \zeta(C+3D)$$

$$b = -(B-3D)$$

$$c = -\zeta(D+1)$$

Hence, the *SUCTRE* becomes synonymous with the *Quadratic Equation*:

$$\zeta(C+3D)\tan^2\theta - (B-3D)\tan\theta - \zeta(D+1) = 0 \qquad \text{[Ref. Equation 30]}$$

$$ax^2 + bx + c = 0$$

Therefore, it too can be relegated to the *geometric mapping process* specified in *Section 2.3*.

Section 18.3 determines that the *tan θ to ζ Linearity Expression* generally is depicted as a *Category 2 equation type* because its *left-hand member* is usually *irrational.*

$$\tan\theta = -(\frac{J}{F})\zeta \qquad \text{[Ref. Equation 48]}$$

It maps out a *straight line* of the form $y = mx + b$ such that:

- o The slope "m" is equal to -J/F and
- o The y-intercept 'b', is equal to zero

Hence, $\tan\theta$ becomes the resulting *ordinate value* for any and all x-axis values of ζ which may be represented on a straight line of slope -J/F which passes through the origin.

Section 18.4 mentions that ***Generalized Cubic Equations*** which express α=1 qualify either as *Category 1* or *Category 3 equation types*, depending upon the nature of their *remaining coefficients.*

Two specific sets of *Generalized Cubic Equations* are afforded along with their associated *Quadratic Equation reductions*. *One set* exhibits only *coefficients* which are *rationally-based (Ref. Section 9.1)*, while the last two *coefficients* of the *other set* are determined to be *completely cubic irrational*. Both *sets of equations* are established by selecting a *specific value* of z_R as follows:

$$z_R = R\tan\theta = \tan\theta_R = -\frac{1}{\zeta} = -\frac{1}{\tan(3\theta)} = -\frac{1}{\sqrt{3}}$$

$$\theta_R = \arctan(-\frac{1}{\sqrt{3}}) = -30^o$$

$$3\theta = \arctan\sqrt{3} = 60^o = \theta_R + \theta_S + \theta_T$$

$$= -30^o + \theta_S + \theta_T$$

$$90^o = \theta_S + \theta_T$$

Hence, θ_S and θ_T are *complementary* to one another such that,

$$\tan\theta_S = \frac{1}{\tan\theta_T}$$

- When θ_S is selected specifically as 45°, θ_T becomes 45° also, such that:

$$\tan 45^o = \frac{1}{\tan 45^o} = 1 = z_S = z_T$$

Where,

$$\beta = -(z_R + z_S + z_T) = -(-1/\zeta + 1 + 1) = -(2 - 1/\zeta)$$
$$\gamma = z_R(z_S + z_T) + z_S z_T = (-1/\zeta)(1+1) + 1(1) = -2/\zeta + 1$$
$$\delta = -z_R z_S z_T = -1[-(1/\zeta)(1)(1)] = 1/\zeta$$

- When θ_S is selected as a specific *cubic irrational* value of 20°:

$$z_S = \tan\theta_S = \frac{1}{\tan\theta_T} = \frac{1}{z_T} = \tan 20^o = 0.363970234$$
$$z_T = \tan\theta_T = \frac{1}{\tan\theta_S} = \frac{1}{z_S}$$

Where,

$$\beta = -(z_R + z_S + z_T) = -(-1/\zeta + \tan\theta_S + \frac{1}{\tan\theta_S}) = -(\tan\theta_S + \frac{1}{\tan\theta_S} - 1/\zeta)$$
$$\gamma = z_R(z_S + z_T) + z_S z_T = -1/\zeta(\tan\theta_S + \frac{1}{\tan\theta_S}) + \tan\theta_S(\frac{1}{\tan\theta_S}) = 1 - 1/\zeta(\tan\theta_S + \frac{1}{\tan\theta_S})$$
$$\delta = -z_R z_S z_T = -1(-(1/\zeta)(\tan\theta_S)(\frac{1}{\tan\theta_S}) = 1/\zeta$$

The resulting compilation is given below:

Category 1 Equation Type Sets	***Category 3 Equation Type Sets***
$z^3 - (2 - \frac{1}{\zeta})z^2 + (1 - \frac{2}{\zeta})z + \frac{1}{\zeta} = 0$	$z^3 - (\tan\theta_S + \frac{1}{\tan\theta_S} - \frac{1}{\zeta})z^2 + [1 - (\frac{1}{\zeta})(\tan\theta_S + \frac{1}{\tan\theta_S})]z + \frac{1}{\zeta} = 0$
$z^3 - 1.422649731z^2 - 0.154700538z + 0.577350269 = 0$	$z^3 - 2.534097385z^2 - 0.79639514z + 0.577350269 = 0$
$z^2 - (z_R + z_S)z + z_R z_S = 0$	$z^2 - (z_R + z_S)z + z_R z_S = 0$
$z^2 - (1 - \frac{1}{\zeta})z - \frac{1}{\zeta} = 0$	$z^2 - (\tan\theta_S - \frac{1}{\zeta})z - \frac{\tan\theta_S}{\zeta} = 0$
$z^2 - 0.42264973z - 0.577350269 = 0$	$z^2 + 0.213380034z - 0.21038312 = 0$

Both Category 1 and *Category 3 reduced Quadratic Equations* shown in the *third row* of the above table may be operated upon via the *Euclidean Quadratic Mapping* process of Section 2.3. Hence, a *compass and straight edge operation* can be applied ***without reservation*** upon given *Quadratic Equations whose coefficients* are either purely *rationally-based* lengths, or a combination thereof. That's because once ***cubic irrational lengths*** become specified as *Quadratic Equation coefficients*, their *respective roots* can be determined via *Euclidean constructions based upon* such *presented lengths*.

Section 18.5 concerns itself with ***Angle Trisector Triangles*** that feature included angles of $\alpha - \phi$ and $3\alpha + \phi$ under specific circumstances when:

$$\tan\phi = \tan^3\alpha$$

They enable *(Ref. Figure 43, triangle AEF)*:

- Angles of $\alpha - \phi$ to be *geometrically constructed* from *given*, or *known* angles of $3\alpha + \phi$, thereby permitting a *geometric determination* of constituent angles α, 3α and ϕ
- *Rationally-based and cubic irrational length combinations to coexist* within *single triangles*
- Mathematical association of such lengths via *Complex Quadratic Equation 50*, as depicted in *two forms* below:

$$a^2 - [2r\cos(3\alpha+\phi)]a - 8r^2 = 0$$

$$r^2 + [\frac{a\cos(3\alpha+\phi)}{4}]r - \frac{a^2}{8} = 0 \qquad \textit{[Ref. Equation 50]}$$

As indicated, both forms exhibit *first term coefficients* with respective values of unity. Hence, equations of either form <u>cannot</u> be depicted as *Category 2 equation types*. Examples for the remaining *equation types* are specified below:

r	cos(3α+ϕ)	COEFFICIENTS: A	COEFFICIENTS: B	COEFFICIENTS: C	EQUATION TYPE: $Ax^2+Bx+C=0$ For: $x_{ABOVE}=a$ $a^2-[2r\cos(3\alpha+\phi)]a-8r^2=0$	a	EQN. CAT.
2 *(R-B)*	-23/12 *(Rat.-based)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Rat.-based)*	$-8r^2$ *(R-B)*	$a^2+\frac{23}{3}a-32=0$	3 *(Rat.-based)*	1
cos20° *(Trans.)*	3/4 *(Rat.-based)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Cubic irrational)*	$-8r^2$ *(Trans.)*	$a^2-1.409538961a-7.064177772=0$	3.454474499 *(Trans.)*	3
cos20° *(Trans.)*	-10.807924 *(Trans.)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Cubic irrational)*	$-8r^2$ *(Trans.)*	$a^2+20.31225391a-7.064177772=0$	sin20° *(Trans.)*	3
cos20° *(Trans.)*	1/ cos20° *(Trans.)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Rat.-based)*	$-8r^2$ *(Trans.)*	$a^2-2a-7.064177772=0$	3.839749597 *(Trans.)*	3
6 *(R-B)*	-70.142803 *(Trans.)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Cubic irrational)*	$-8r^2$ *(R-B)*	$a^2+841.7136472a-288=0$	sin20° *(Trans.)*	3
sin20° *(Trans.)*	-4.71403069 *(Trans.)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Cubic irrational)*	$-8r^2$ *(Trans.)*	$a^2+3.224586908a-0.935822227=0$	$2-\sqrt{3}$ *(Rat.-based)*	3

a	cos(3α+ϕ)	COEFFICIENTS: A	COEFFICIENTS: B	COEFFICIENTS: C	EQUATION TYPE: $Ax^2+Bx+C=0$ For: $x_{ABOVE}=r$ $r^2+[\frac{a\cos(3\alpha+\phi)}{4}]r-\frac{a^2}{8}=0$	r	EQN. CAT.
3 *(Rat.-based)*	-23/12 *(Rat.-based)*	1 *(R-B)*	$a\cos(3\alpha+\phi)/4$ *(Rat.-based)*	$-a^2/8$ *(R-B)*	$r^2-\frac{23}{16}r-\frac{9}{8}=0$	2 *(R-B)*	1
3.454474499 *(Trans.)*	¾ *(Rat.-based)*	1 *(R-B)*	$a\cos(3\alpha+\phi)/4$ *(Cubic irrational)*	$-a^2/8$ *(Trans.)*	$r^2+0.647713968r-1.491674258=0$	cos20° *(Trans.)*	3
sin20° *(Trans.)*	-10.807924 *(Trans.)*	1 *(R-B)*	$a\cos(3\alpha+\phi)/4$ *(Cubic irrational)*	$-a^2/8$ *(Trans.)*	$r^2-0.924131976r-0.014622222=0$	cos20° *(Trans.)*	3
3.839749597 *(Trans.)*	1/ cos20° *(Trans.)*	1 *(R-B)*	$a\cos(3\alpha+\phi)/4$ *(Cubic irrational)*	$-a^2/8$ *(Trans.)*	$r^2+1.021544043r-1.892959621=0$	cos20° *(Trans.)*	3
sin20° *(Trans.)*	-70.142803 *(Trans.)*	1 *(R-B)*	$a\cos(3\alpha+\phi)/4$ *(Cubic irrational)*	$-a^2/8$ *(Trans.)*	$r^2-5.997562963r-0.014622222=0$	6 *(R-B)*	3
$2-\sqrt{3}$ *(Rat.-based)*	-4.71403069 *(Trans.)*	1 *(R-B)*	$a\cos(3\alpha+\phi)/4$ *(Cubic irrational)*	$-a^2/8$ *(R-B)*	$r^2-0.315780179r+\frac{4\sqrt{3}-7}{8}=0$	sin20° *(Trans.)*	3

Equation equality is *preserved* when the sum of the *three terms* in *Equation 50* equals zero. As demonstrated in the *two* additional tables shown below, this can be achieved only when either:

- All three *terms* are *rationally based*
- All three *terms* are *cubic irrational*
- Two of the *terms* are *cubic irrational* such that they sum to the value of a *rationally-based* third term

r	**EQUATION TYPE:** $Ax^2 + Bx + C = 0$ ***For*** $x_{ABOVE} = r$ **:** $r^2 + [\frac{a\cos(3\alpha+\phi)}{4}]r - \frac{a^2}{8} = 0$	TERMS		
		Ax^2	Bx	C
2 *(Rat.-based)*	$r^2 - \frac{23}{16}r - \frac{9}{8} = 0$	4 *(Rat.-based)*	-23/8 *(Rat.-based)*	-9/8 *(Rat.-based)*
cos20° *(Cubic irrational)*	$r^2 + 0.647713968r - 1.491674258 = 0$	0.883022221 *(Cubic irrational)*	0.608652026 *(Cubic irrational)*	-1.491674258 *(Cubic irrational)*
cos20° *(Cubic irrational)*	$r^2 - 0.924131976r - 0.014622222 = 0$	0.883022221 *(Cubic irrational)*	-0.868399998 *(Cubic irrational)*	-0.014622222 *(Cubic irrational)*
cos20° *(Cubic irrational)*	$r^2 + 1.021544043r - 1.892959621 = 0$	0.883022221 *(Cubic irrational)*	+0.959937399 *(Cubic irrational)*	-1.892959621 *(Cubic irrational)*
6 *(Rat.-based)*	$r^2 - 5.997562963r - 0.014622222 = 0$	36 *(Rat.-based)*	-35.98537778 *(Cubic irrational)*	-0.014622222 *(Cubic irrational)*
sin20° *(Cubic irrational)*	$r^2 - 0.315780179r + \frac{4\sqrt{3}-7}{8} = 0$	0.116977778 *(Cubic irrational)*	-0.108003182 *(Cubic irrational)*	-0.008974596 *(Rat.-based)*

a	**EQUATION TYPE:** $Ax^2 + Bx + C = 0$ ***For:*** $x_{ABOVE} = a$ $a^2 - [2r\cos(3\alpha+\phi)]a - 8r^2 = 0$	TERMS		
		Ax^2	Bx	C
3 *(Rat.-based)*	$a^2 + \frac{23}{3}a - 32 = 0$	9 *(Rat.-based)*	23 *(Rat.-based)*	-32 *(Rat.-based)*
3.454474499 *(Cubic irrational)*	$a^2 - 1.409538961a - 7.064177772 = 0$	11.93339406 *(Cubic irrational)*	-4.869216396 *(Cubic irrational)*	-7.064177772 *(Cubic irrational)*
sin20° *(Cubic irrational)*	$a^2 + 20.31225391a - 7.064177772 = 0$	0.116977778 *(Cubic irrational)*	6.947199994 *(Cubic irrational)*	-7.064177772 *(Cubic irrational)*
3.839749597 *(Cubic irrational)*	$a^2 - 2a - 7.064177772 = 0$	14.74367697 *(Cubic irrational)*	-7.679499194 *(Cubic irrational)*	-7.064177772 *(Cubic irrational)*
sin20° *(Cubic irrational)*	$a^2 + 841.7136472a - 288 = 0$	0.116977778 *(Cubic irrational)*	287.8830223 *(Cubic irrational)*	-288 *(Rat.-based)*
$2-\sqrt{3}$ *(Rat.-based)*	$a^2 + 3.224586908a - 0.935822227 = 0$	0.071796769 *(Rat.-based)*	0.864025456 *(Cubic irrational)*	-0.935822227 *(Cubic irrational)*

Section 18.5 affords a *numerical example* for:

$3\alpha + \phi = 119.4335543^o$

Side $\overline{A'E} = a$ is of *rationally-based length* $\sqrt{3}/2$

Its other two sides are to be expressed by the following two *cubic irrational lengths*:

$\overline{FE} = r = \tan 20^o = 0.363970234$

$\overline{A'F} = 3r = 3\tan 20^o = 1.091910703$ *[Ref. Figure 43]*:

Hence, a *cubic irrational number pair* may be determined from another completely *independent cubic irrational number, such as cos* $(3\alpha+\phi)$, in consonance with a *given rationally-based* number, such as $\sqrt{3}/2$.

Figure 44 depicts a *geometric construction* of the *cubic irrational length* $r = \tan 20^o = 0.363970234$. Such length was *geometrically constructed* using the *Euclidean mapping process* specified in *Section 2.3*, and is premised upon the *coefficients* for *Equation 50* reiterated *below*:

$a^2 - 2ar\cos(3\alpha+\phi) - 8r^2 = 0$	*[Ref. Equation 50]*

$$-8a^2 + 16a\cos(3\alpha+\phi)r + r^2 = 0$$

$$r^2 + 16a\cos(3\alpha+\phi)r - 8a^2 = 0$$

$$\tan^2 20^o - (0.106394226)\tan 20^o - \frac{3}{32} = 0$$

Figure 45 and *Figure 46* demonstrate additional *geometric construction* that is considered necessary in order to achieve the above rendering.

In conclusion, it is contended that *cubic irrational numbers, or cubic irrational lengths,* appear as *pairs or conjugates* in *Complex Quadratic Equations* where one may be determined via the other.

Section 18.6 portrays the *Cosine Circle*, a novel *geometric construction* which locates *root sets* by a simple two step process which consists of

1) Rotating an *inscribed equilateral triangle* about its *origin* until its vertices align with designated *angle sets* of θ, $\theta + 120^o$, and $\theta + 240^o$ *(Ref. Figure 47)*; and
2) *Dropping perpendiculars* about select points.

The *Cosine Circle* applies to the following *equation formats*:

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta) \qquad \text{[Ref. Equation 1]}$$

With roots *(Ref. Section 2.4.1)*:

$$x_1 = \cos\theta$$

$$x_2 = \cos(\theta + 120^o)$$

$$x_3 = \cos(\theta + 240^o)$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta) \qquad \text{[Ref. Equation 2]}$$

With roots *(Ref. Section 2.4.2)*:

$$y_1 = \sin\theta$$

$$y_2 = \sin(\theta + 120^o)$$

$$y_3 = \sin(\theta + 240^o)$$

$$\tan^3\theta = 3\tan\theta - \tan(3\theta)(1 - 3\tan^2\theta) \qquad \text{[Ref. Equation 3]}$$

With roots *(Ref. Section 2.4.3)*:

$$z_1 = \tan\theta$$
$$z_2 = \tan(\theta + 120^o)$$
$$z_3 = \tan(\theta + 240^o)$$

Such *roots* may be applied in orderly fashion via *geometric construction* in order to determine their respective 3θ *trigonometric counterparts* as defined below:

$$x_1 x_2 x_3 = \frac{\cos(3\theta)}{4}$$ *[Ref. Equation 5]*

$$x_1 + x_2 + x_3 = 0$$ *[Ref. Equation 6]*

$$x_1 x_2 + x_1 x_3 + x_2 x_3 = -\frac{3}{4}$$ *[Ref. Equation 7]*

$$y_1 y_2 y_3 = -\frac{\sin(3\theta)}{4}$$ *[Ref. Equation 8]*

$$y_1 + y_2 + y_3 = 0$$ *[Ref. Equation 9]]*

$$y_1 y_2 + y_1 y_3 + y_2 y_3 = -3/4$$ *[Ref. Equation 10]*

$$z_1 z_2 z_3 = -\tan(3\theta)$$ *[Ref. Equation 11]*

$$z_1 + z_2 + z_3 = 3\zeta$$ *[Ref. Equation 12]*

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = -3$$ *[Ref. Equation 13]*

Examples of *Cosine Circle* supporting equations can be found for each of the *category equation types* enumerated above.

For instance, it is shown that the *following equation* can qualify as a *Category 1 Equation Type* when each of its *coefficients* is considered to encompass, or equal the value of its *entire respective term*, under the specific condition when $3\theta = 45^o$:

$$x_1 + x_2 + x_3 = 0$$ *[Ref. Equation 6]*

$$\cos\theta + \cos(\theta + 120^o) + \cos(\theta + 240^o) = 0$$

$$\frac{\sqrt{\sqrt{3}+2}}{2} - \frac{\sqrt{3}+1}{2\sqrt{\sqrt{3}+2}} - \frac{1}{2\sqrt{\sqrt{3}+2}} = 0$$

Hence, each *entire respective term* is *rationally-based* in itself. Once each is considered to be that term's coefficient, the equality is verified to hold as follows:

$$\frac{(\sqrt{3}+2) - (\sqrt{3}+1) - 1}{2\sqrt{\sqrt{3}+2}} = 0$$

$$0 = 0$$

In *Section 18* above, many instances are afforded whereby *rationally-based* and *cubic irrational* lengths, evident within *Quadratic Equations*, are portrayed ***geometrically.***

The *Quadratic Formula* solely is responsible for this! It serves as a *known bastion* or so-called *last frontier* that can be used to *properly interrelate* two completely *independent branches of mathematics* – namely, ***algebra*** and ***geometry***!

SECTION 19

Two forms of the *Generalized Cubic Equation (GCE)* are the:

1) *3θ Cubic Equation* $z^3 - 3\zeta z^2 - 3z + \zeta = 0$, now to be described herein as a ***primary GCE***; and
2) Those whose "R" values are equal to unity, now to be described as ***secondary, independent GCE's***.

A ***simultaneous resolution*** is considered to occur when *pairs* of such types of *Generalized Cubic Equations* become *algebraically manipulated* with respect to a *common root* z_R which they *both* are considered to share.

Quadratic Equation ***reductions*** result, which thereafter can be charted via the *geometric mapping process* stipulated in *Section 2.3*.

Such ***primary GCE*** is of particular value because:

- Its *coefficients* are either known rational values of *unity* and *-3*, or discernable in terms of any postulated value of $\zeta = \tan(3\theta)$
- Its root $z_R = (R)\tan\theta = (1)\tan\theta = \tan\theta$

Such <u>second</u>, *independent Generalized Cubic Equation*, one which possesses the very same ***common root*** $z_R = \tan\theta$, then simply is to retain the same *coefficient structure* as the *GCE* itself, stipulated as follows:

$$(1)z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

A mathematical substitution becomes possible since both above equations share a ***common root*** $\boldsymbol{z_R}$ whereby such *primary GCE* can be reformatted as:

$$z_R^{\ 3} - 3\zeta z_R^{\ 2} - 3z_R + \zeta = 0$$

$$z_R^{\ 3} = 3\zeta z_R^{\ 2} + 3z_R - \zeta$$

Substitution from above into such *second, independent Cubic Equation* renders the *reduced Quadratic Equation*:

$$(3\zeta z_R^{\ 2} + 3z_R - \zeta) + \beta z_R^{\ 2} + \gamma z_R + \delta = 0$$

$$(3\zeta + \beta)z_R^{\ 2} + (3 + \gamma)z_R + (\delta - \zeta) = 0$$

Now, since,

$$\theta_S + \theta_T = 2\theta$$

$$\tan(\theta_S + \theta_T) = \tan(2\theta)$$

$$\frac{\tan\theta_S + \tan\theta_T}{1 - \tan\theta_S \tan\theta_T} = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\frac{z_S + z_T}{1 - z_S z_T} = \frac{2z_R}{1 - z_R{}^2}$$

$$(z_S + z_T)(1 - z_R{}^2) = 2z_R(1 - z_S z_T)$$

$$(z_S + z_T)(1 - z_R{}^2) = 2z_R + 2\delta$$

$$-2\delta = 2z_R + (z_S + z_T)(z_R{}^2 - 1)$$

$$= 2z_R - (\beta + z_R)(z_R{}^2 - 1)$$

$$= (3z_R - z_R{}^3) - \beta(z_R{}^2 - 1)$$

Such that,

$$\zeta = \frac{\delta - \beta}{1-\gamma} \qquad \textit{[Ref. Equation 36]}$$

$$\zeta(1-\gamma) = \delta - \beta$$

$$-\delta = -\beta - \zeta(1-\gamma)$$

$$-2\delta = -2\beta - 2\zeta(1-\gamma)$$

Via substitution,

$$-2\beta - 2\zeta(1-\gamma) = (3z_R - z_R{}^3) - \beta(z_R{}^2 - 1)$$

$$2\zeta(1-\gamma) + 2\beta = -(3z_R - z_R{}^3) + \beta(z_R{}^2 - 1)$$

$$2\zeta - 2\zeta\gamma + 2\beta = -\zeta(1 - 3z_R{}^2) + \beta(z_R{}^2 - 1)$$

$$3\zeta - 2\zeta\gamma + 3\beta = (3\zeta + \beta)z_R{}^2$$

$$3(\zeta + \beta) - 2\zeta\gamma = (3\zeta + \beta)z_R{}^2$$

The above *right-hand term* has exactly the same value as the first term of the *left-hand member* listed in *Generalized Cubic Equation* reduction shown above and restated below:

$$(3\zeta + \beta)z^2 + (3+\gamma)z + (\delta - \zeta) = 0$$

Substitution renders:

$$3(\zeta + \beta) - 2\zeta\gamma + (3+\gamma)z_R + (\delta - \zeta) = 0$$

$$(3+\gamma)z_R = 2\zeta\gamma - 3(\zeta+\beta) - (\delta - \zeta)$$

$$z_R = \frac{2\zeta\gamma - 3(\zeta+\beta) - (\beta - \zeta\gamma)}{3+\gamma}$$

Therefore, the ***Coefficient Structure of a Second, Independent GCE for R=1*** is as follows:

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

The entire *process* of *simultaneously resolving GCE pairs* which are linked by their ***common root*** $z_R = (R)\tan\theta = (1)\tan\theta = \tan\theta$ consists of:

1) *Identifying* an *angle 3θ* for analysis;
2) *Geometrically constructing* its *tangent* $\zeta = \tan(3\theta)$;
3) *Specifying* its *3θ Cubic Equation*; and lastly
4) *Specifying* an *associated second, independent GCE*.

Respective coefficients of such *second, independent* GCE can be determined in accordance with *Equation 51* as follows:

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

1) $z_R = \tan\theta$ is calculated *trigonometrically* from $\zeta = \tan(3\theta)$
2) A *designated value* of β becomes arbitrarily assigned
3) *Coefficient* γ then becomes readily calculated
4) *Remaining coefficient δ* becomes calculated via *Equation 36* as follows:

$$\zeta = \frac{\delta-\beta}{1-\gamma} \qquad \textit{[Ref. Equation 36]}$$

$$\zeta(1-\gamma)+\beta = \delta$$

Above a singular value for the *unknown coefficient* γ is readily obtained by first ascribing ***properly associated*** $\zeta = \tan(3\theta)$ and $z_R = \tan\theta$ *trigonometric values*, and thereafter assigning an arbitrary value to β.

However, when not relying upon the fact that $z_R = \tan\theta$ can be *trigonometrically* determined from $\zeta = \tan(3\theta)$, such *common root value* instead must be ascertained from the two remaining *unknown values β* and γ *(Ref. Equation 51)*.

This can be accomplished only when such ***correct*** singular value of γ becomes interposed into *Equation 51* with respect to each and every specific value of $\zeta = \tan(3\theta)$ and arbitrarily assigned value of β which also become applied to it.

Unfortunately, in most cases, without having ***advance knowledge*** of such $z_R = \tan\theta$ to $\zeta = \tan(3\theta)$ *trigonometric relationship*, it becomes impossible to distinguish the proper value of γ that should become inserted in the first place.

In other words, γ then would become distinguishable by *Equation 51* only after properly associated values of ζ and the ***unknown common root*** $\boldsymbol{z_R}$, along with an arbitrarily assigned value of β *first* become disclosed.

- ***More specifically restated***:
  Aforehand knowledge of such *common root value* z_R, would be needed in order to enable determination of the respective values of ***coefficients*** which belong to, or fully characterize such coterie of *second*, *independent GCE's*
- ***Even more fully explained***:
  A *second*, *independent GCE*, considered to be a *Cubic Equation* whose coefficients could be fed into the algebraic linear *Equation 51* for purposes of obtaining a ***common root value*** $\boldsymbol{z_R}$ that, in turn, could be operated upon via *geometric construction* in order to produce a *trisected angle θ*, cannot be determined without having ***aforehand knowledge*** of such ***common root value*** $\boldsymbol{z_R}$ in the first place

Such *preponderance* poses an *insurmountable difficulty* or *unfathomable discontinuity* for the *Euclidean process* which ***must*** be told exactly which *coefficient values* are to applied in order to *geometrically construct* a ***common root*** $\boldsymbol{z_R}$.

Therefore, *it is concluded* that when a *coefficient structure* for a *second, independent GCE*:

1) Can be determined without gaining ***aforehand knowledge*** of the value of its ***common root*** $\boldsymbol{z_R}$, then such equation *can* be used to *reduce* its *associated 3θ Cubic Equation* into *quadratic form*, thereby enabling a ***simultaneous resolution*** via the *geometric mapping process specified in Section 2.3*; which in turn enables the depiction of an *angle θ* which represents a bonafide *trisector* for any *given, or assigned angle 3θ (Ref. Section 20)*; or
2) Cannot be determined without gaining ***aforehand knowledge*** of the value of its ***common root*** $\boldsymbol{z_R}$, then such equation *cannot* be fed into *linear Equation 51* for purposes of obtaining a *common root value* z_R that, in turn, could have been operated upon via *geometric construction* in order to produce a *trisected angle θ*.

This second above premise is demonstrated for the case when the ***common root*** $\boldsymbol{z_R}$ and $\zeta = \tan(3\theta)$ are both ***rational*** as follows;

Where,

$$\zeta = \tan(3\theta) = \frac{13}{9}$$

$$3\theta = 55.30484647^o$$

$$\theta = 18.43494882^o$$

$$z_R = \tan\theta = \frac{1}{3}$$

The resulting *3θ Cubic Equation* is as follows:

$$z_R^{\ 3} - 3\zeta z_R^{\ 2} - 3z_R + \zeta = 0$$

$$z_R^{\ 3} - 3(13/9)z_R^{\ 2} - 3z_R + 13/9 = 0$$

$$z_R^{\ 3} - (13/3)z_R^{\ 2} - 3z_R + 13/9 = 0$$

As such, it becomes obvious that:

- The *coefficients* contained in the *3θ Cubic Equation* presented above, in addition to the value of $\zeta = \tan(3\theta) = 13/9$, all represent *rational lengths* and, hence, can be ***geometrically constructed*** simply by means of applying a *straightedge* and *compass* alone.

  This is because they all *stem* from any given or assigned length of unity *(Ref. Section 9.1)*.
- The ***common root value*** $z_R = \tan\theta = 1/3$ also is a *rational length;* whereby, portrayal of the *trisected angle θ*, in this particular case, also rather easily could be produced via ***geometric construction*** using only *Euclidean tools*.

However, no ***geometric construction method*** exists which can determine $z_R = 1/3$ when only a known *value of* $\zeta = \tan(3\theta) = 13/9$ is supplied or *given* in the first place; thereby thwarting any attempts to perform *Euclidean trisection*.

By introduction of *Equation 51*, such above stated impossibility is explained *mathematically* as follows:

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

$$\frac{1}{3} = \frac{3(\frac{13}{9})(\gamma-1)-4\beta(\frac{3}{3})}{3+\gamma}$$

$$3+\gamma = 13(\gamma-1)-12\beta$$

For the specific case when $\beta = 0$:

$$3+\gamma = 13(\gamma-1)-12(0)$$

$$3+\gamma = 13\gamma-13$$

$$16 = 12\gamma$$

$$4/3 = \gamma$$

$$\delta = \zeta(1-\gamma)+\beta$$

$$= \frac{13}{9}[1(\frac{3}{3})-\frac{4}{3}]+0$$

$$= -\frac{13}{9}(\frac{1}{3})$$

$$= -13/27$$

As such, one bonafide *second independent GCE* for $R=\alpha=1$ is:

$$\alpha z^3+\beta z^2+\gamma z+\delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3+\frac{4}{3}z-\frac{13}{27}=0$$

However, the above determination could not have been rendered without first having received ***aforehand knowledge*** of the *common rational* **common root value** $z_R = 1/3$.

Quite obviously, this algebraic approach is not permitted when attempting to *trisect* an angle via *Euclidean means*!

Equation 51 may be applied in a variety of ways. For instance:

- It validates the *3θ Cubic Equation* by substituting the value of its *third term coefficient* $\gamma = -3$ into *Equation 51* as follows:

$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma}$	*[Ref. Equation 51]*

$$z_R(3+\gamma) = 3\zeta(\gamma-1)-4\beta$$

$$z_R(3-3) = 3\zeta(-3-1)-4\beta$$

$$0 = 3\zeta(-4)-4\beta$$

$$4\beta = 3\zeta(-4)$$

$$\beta = -3\zeta$$

From above:

$$\zeta(1-\gamma)+\beta = \delta$$

$$\zeta[1-(-3)]-3\zeta = \delta$$

$$\zeta = \delta$$

Hence, such *Generalized Cubic Equation* when $\alpha = 1$ reduces to the *3θ Cubic Equation* as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \text{[Ref. Equation 32]}$$

$$z_R{}^3 - 3\zeta z_R{}^2 - 3z_R + \delta = 0 \qquad \text{Q.E.D.}$$

- It validates that *Generalized Cubic Equations* whose ***sub-element*** $R = 1$ contain a root whose value is equal to the negative of its *β coefficient* when $\gamma = +1$ as follows:

$$z_R(3+\gamma) = 3\zeta(\gamma - 1) - 4\beta$$

$$z_R(3+1) = 3\zeta(1-1) - 4\beta$$

$$z_R = -\beta$$

Another *second independent GCE* for $R = 1$ example is featured below to *emphasize that knowledge* of the ***common root value*** z_R is needed ***aforehand*** in order to characterize its respective *coefficient structure*. Given that:

$$\theta_R = \theta$$

$$\theta_S = \theta + 45^o$$

$$\theta_T = \theta - 45^o$$

$$\Sigma = \theta_R + \theta_S + \theta_T = 3\theta$$

Such that,

$$z_R = \tan\theta_R = \tan\theta$$

$$z_S = \tan\theta_S = \tan(\theta + 45^o) = \frac{\tan\theta + 1}{1 - \tan\theta}$$

$$z_T = \tan\theta_T = \tan(\theta - 45^o) = \frac{\tan\theta - 1}{1 + \tan\theta}$$

Where,

$$\beta = -(z_R + z_S + z_T)$$

$$\gamma = z_R(z_S + z_T) + z_S z_T$$

$$\delta = -z_R z_S z_T$$

Therefore, the *second independent GCE* for $R = 1$ and $\alpha = 1$ is determined to be:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \text{[Ref. Equation 32]}$$

$$z_R{}^3 + \beta z_R{}^2 + \gamma z_R + \delta = 0$$

$$z_R{}^3 - \left(\frac{5\tan\theta - \tan^3\theta}{1 - \tan^2\theta}\right) z_R{}^2 + \left[\tan\theta\left(\frac{4\tan\theta}{1 - \tan^2\theta}\right) - 1\right] z_R + \tan\theta = 0$$

Clearly, all the *coefficients* enumerated above are represented as functions of the ***common root*** $z_R = \tan\theta$. Hence, the *coefficient values* of such *Generalized Cubic Equation* cannot be determined without having ***aforehand knowledge*** of its *root* z_R.

Even when *three roots* and *all coefficients* belonging to a *second, independent GCE* for $R = 1$ can be ***geometrically constructed***, *trisecting* an associated *given 3θ angle* still remains intractable. An example of this is provided below:

$$z_R = R\tan\theta = \tan\theta_R = -1/\sqrt{3}$$
$$z_S = S\tan\theta = \tan\theta_S = 1$$
$$z_T = T\tan\theta = \tan\theta_T = 1$$

$$\theta_R = -30^o$$
$$\theta_S = 45^o$$
$$\theta_T = 45^o$$
$$\Sigma = 3\theta = 60^o$$

$$\zeta = \tan(3\theta) = \sqrt{3}$$

$$\theta = 60^o/3 = 20^o$$

$$\tan\theta = 0.363970234$$

For $\alpha = 1$:

$$\beta = -(z_R + z_S + z_T) = -(-\frac{1}{\sqrt{3}} + 1 + 1) = -1.422649731$$

$$\gamma = z_R(z_S + z_T) + z_S z_T = -\frac{1}{\sqrt{3}}(1+1) + (1)(1) = -0.154700538$$

$$\delta = -z_R z_S z_T = -(-\frac{1}{\sqrt{3}})(1)(1) = 0.577350269$$

Then,

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 - 1.422649731z^2 - 0.154700538z + 0.577350269 = 0$$

All above determined *coefficients* can be *constructed* via *Euclidean means* since they all are *rationally-based (Ref. Section 9.1)*; that is, they represent *mathematical combinations* of the *associated GCE root structure* consisting of 1, 1, and $-1/\sqrt{3}$.

However, such *associated GCE* contains ***no*** roots *in common* with its respective *3θ Cubic Equation*. Hence, $R \neq 1$ and such resulting equation cannot qualify as a *second, independent GCE* for $R = 1$. This is evidenced by the *root structure* for each presented below:

3θ Cubic Equation Roots	*Associated GCE Roots*
$z_1 = \tan\theta_R = \tan\theta = 0.363970234$	$z_R = R\tan\theta = -1/\sqrt{3}$
$z_2 = \tan\theta_S = \tan(\theta + 120^o) = -0.839099631$	$z_S = S\tan\theta = 1$
$z_3 = \tan\theta_T = \tan(\theta + 240^o) = 5.67128182$	$z_T = T\tan\theta = 1$

Above, ***associated GCE*** *roots* are represented as *Complex Linear Equations* expressing $\tan\theta$ and respective values of *R*, *S*, and *T*; all unknown terms that cannot be deciphered by *Euclidean* means.

This means that *many* values of *R*, for example, can be arbitrarily introduced, such that *compensating values* of $\tan\theta$ must equal $-1/(\sqrt{3}R)$. Moreover, only *one* unknown value of -1.586256828 for *R* correctly determines $\tan\theta = 0.363970234$.

Using the information provided above, an example *simultaneous reduction* of a *3θ Cubic Equation* with its *second, independent GCE for* $R=1$ is afforded as follows:

$$z_R^{\,3}-(\frac{5\tan\theta-\tan^3\theta}{1-\tan^2\theta})z_R^{\,2}+[\tan\theta(\frac{4\tan\theta}{1-\tan^2\theta})-1]z_R+\tan\theta=0$$

For the particular condition when,

$$3\theta=60^o$$

$$\theta=20^o$$

$$z_R=\tan\theta=0.363970234$$

$$z_R^{\,3}-3\zeta z_R^{\,2}-3z_R+\zeta=0 \qquad \textit{[3θ Cubic Equation]}$$

$$z_R^{\,3}-3(\tan 60^o)z_R^{\,2}-3z_R+(\tan 60^o)=0$$

$$z_R^{\,3}-3\sqrt{3}z_R^{\,2}-3z_R+\sqrt{3}=0$$

$$z_R^{\,3}=3\sqrt{3}z_R^{\,2}+3z_R-\sqrt{3}$$

The *coefficients* for a *second, independent GCE for* $R=1$ become:

$$\beta=-(\frac{5\tan\theta-\tan^3\theta}{1-\tan^2\theta})=-2.042169497 \qquad \textit{[Established above]}$$

$$\gamma=\tan\theta(\frac{4\tan\theta}{1-\tan^2\theta})-1=-0.389185421 \qquad \textit{[Established above]}$$

$$\delta=\tan\theta=0.363970234 \qquad \textit{[Established above]}$$

Hence, this particular associated second, independent *GCE* for $R=1$ and $\alpha=1$ is:

$$\alpha z^3+\beta z^2+\gamma z+\delta=0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3-2.042169497z^2-0.389185421z+0.363970234=0$$

Check,

$$\zeta=\frac{\delta-\beta}{1-\gamma} \qquad \textit{[Ref. Equation 36]}$$

$$=\frac{0.363970234-(-2.042169497)}{1-(-0.389185421)}$$

$$=\frac{2.406139731}{1.389185421}$$

$$\tan 60^o=\sqrt{3}$$

From the *simplified quadratic equation* determined in the derivation of *Equation 51*:

$$(3\zeta+\beta)z_R^{\,2}+(3+\gamma)z_R+(\delta-\zeta)=0$$

$$(3\sqrt{3}-2.042169497)z_R^{\,2}+(3-0.389185421)z_R+(0.363970234-\sqrt{3})=0$$

$$3.153982926z_R^{\,2}+2.610814579z_R-1.368080573=0$$

The resulting resolution follows:

$$z_1;z_2=\frac{1}{2a}(-b\pm\sqrt{b^2-4ac})$$

$$=\frac{1}{2(3.153982926)}[-2.610814579\pm\sqrt{(2.610814579)^2+4(3.153982926)(1.368080573)}]$$

$$=0.363970234;-1.191753593$$

$$=\tan 20^o;-\frac{1}{\tan 40^o}$$

$$=\tan\theta;-\frac{1}{\tan(2\theta)}$$

In summary, Equation 51 depicts a ***remarkable portrayal*** of the very manner in which an *unknown* **common root z_R** manifests itself via inextricable linkage to *modifying coefficients*.

Other *well known* equation formats which relate *root structures* to their *coefficients* are as indicated below:

- The *Quadratic Formula* relates its *roots* to *respective coefficients* via the *Quadratic Formula* as follows:
  Where $ax^2 + bx + c = 0$,

  $$x_1; x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The *Generalized Cubic Equation* relates its *roots* to *respective coefficients* as follows when $\beta^2 = 3\alpha\gamma$ and $\alpha = 1$:

  $$z_R = \frac{-\beta + \sqrt[3]{\beta^3 - 27\delta}}{3}$$ *[Ref. Section 13.2]*

In conclusion, such *associations* between *equation coefficients* and their *intrinsic root structures* are best *characterized* by *mathematical interpretations* of their inherent ***RST Spreads.***

Section 20

To reiterate what clearly has been asserted many times over during the past years: An angle most certainly cannot be trisected solely via Euclidean means! More specifically stated, that is to say it is impossible to trisect an angle, no matter what its size, when only a straightedge and compass are permitted to act upon it.

In response to the caveat that certain angles can be trisected, let it be said that such actions cannot be achieved solely by Euclidean means, but only when otherwise introducing extraneous information into such famous trisection problem, thereby corrupting it and, in so doing, enabling entirely different problem types to become solved.

In this sense, extraneous information is considered to entail any aforehand knowledge which can be derived from either algebraic determinations, or geometric applications other than those where a straightedge and compass become applied to an angle of given magnitude.

Section 20.1 examines an instance when *Equation 51* becomes invoked for the condition when $\beta = \gamma = 0$.

In this case, the calculations provided below reveal that the tangent value (z_R) of a particular trisector is equal to the negative value of the tangent $(-\zeta)$ of an angle that amounts to exactly three times its size:

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma}$$ *[Ref. Equation 51]*

$$= \frac{3\zeta(0-1)-4(0)}{3+0}$$

$$= -\zeta$$

$$z_R = -(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}) = R\tan\theta = (1)\tan\theta = \tan\theta$$ *[Ref. Equation 3]*

Cross multiplication yields a *reduced Quadratic Equation* whose *unknown, tan θ*, may be resolved via ***Euclidean*** *mapping process* stipulated in *Section 2.3*, mathematically portrayed as follows:

$$\tan\theta(3-\tan^2\theta) = \tan\theta(3\tan^2\theta - 1)$$

$$3-\tan^2\theta) = 3\tan^2\theta - 1$$

$$1 = \tan^2\theta$$

$$\pm 1 = \tan\theta$$

$$\arctan(+1);\arctan(-1) = \arctan\theta_1;\arctan\theta_2$$

$$45^o;135^o = \theta_1;\theta_2$$

$$3(45^o);3(135^o) = 3\theta_1;3\theta_2$$

$$135^o;405^o = 3\theta_1;3\theta_2$$

$$\tan 135^o;\tan(45+360)^o = \zeta_1;\zeta_2$$

$$\mp 1 = \zeta$$

$$\pm 1 = -\zeta = z_R$$

Since both such common root $z_R = \pm 1$, and $\zeta = \tan(3\theta) = \mp 1$ exhibit rationally-based tangent values, the angle θ then could be drawn to represent a true trisector of a given angle 3θ whose magnitude would be either 135°, or 405° = (360o + 45°) = 45°. However, such *geometric construct* would not constitute an act of trisecting an angle solely by the use of a straightedge and compass.

Section 20.2 commissions a *1994 never before published copyright* which, although today appears to be of rather innocuous intent, still apparently manages to be the *first on record* to articulate an ability to achieve bonafide ***Euclidean trisection*** predicated upon a method of ***repeated bisections***.

Therein, a *series of bisections* contrived purely of *compass* and *straightedge* operations is applied in to achieve such *actual trisection* of a *given angle 3θ*.

Unfortunately it requires an infinite *number of iterations* to produce an exact solution. However, after *twenty of such iterations*, a precision of better than *one in a million* would be obtained *(Ref. Table 34)*.

Perhaps this method has received very little attention over the years because it doesn't render an *immediate solution*. Or, quite possibly, it just never was considered before, as it relates to the *mathematics* which governs *geometric progression*, also presented below.

Denoted as "s", the *sum* of an *infinite number of terms* expressed in a *geometric progression*, or series of terms connected by a constant multiplier, is:

$$s = \frac{f}{1-m}$$ *(Ref. second footnote of Section 20)*

Where

- o "f" represents its first term
- o "m" represents a *common ratio* between its terms

When its first term is equal to *3θ*, and m is set equal to -1/2, "s" is found to be equal to 2θ as follows:

$$s = \frac{3\theta}{1-(-1/2)} = 2\theta$$

Since an angle *2θ* can be bisected to produce one of *θ*, this above analysis evidences that any given angle 3θ can be trisected by a *series of Euclidean bisections* conducted in the sequence specified in *Figure 48*.

A *geometric progression* consisting of "n" terms is determined by constantly multiplying each successive term by -1/2 as follows:

$$s = 3\theta - \frac{3\theta}{2} + \frac{3\theta}{4} - \frac{3\theta}{8} + \frac{3\theta}{2^{n-2}} - \frac{3\theta}{2^{n-1}}$$

Each of these above designated terms is located within the circle illustrated in *Figure 48*. They represent *swings of* specified angles from a given *start point* where counterclockwise movement is notated by a positive swing. The location of each respective *end point* is identified outside of the circle. Each location represents a summation of the above specified *geometric progression* for the quantity of terms being depicted. Such respective calculations are afforded in *Table 34*.

Section 20.2 examines the nuances associated with attempting to geometrically construct Equation 1, as denoted below (Ref. Figure 49):

$\cos^3\theta = ¾ \cos\theta + ¼ \cos(3\theta)$ [*Ref. Equation 1*]

<u>SECTION 21</u>

This section investigates the role which *cube roots* play in attempting the impossible act of performing *Euclidean trisection*. The discussion begins by affirming that *root set values* belonging to ***Quadratic Equations*** of the form $ax^2 + bx + c = 0$ can be *algebraically determined* solely from their *coefficient structures* through the *Quadratic Formula* shown below, and furthermore attesting to the fact that they can be ***geometrically constructed*** by means of performing the *Euclidean mapping procedure* stipulated in Section 2.3; whereby the values of their *coefficients* would become represented by lengths of given size.

$$x_1; x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Such digression continues by noting that some *mathematicians*, upon becoming inspired by such ***coefficient driven*** realization, naturally might try to identify some hidden, unknown ***inextricable geometric linkage*** that could associate solely ***rational coefficients*** inherent within *Generalized Cubic Equation formats* to their intrinsic ***cubic irrational root set*** counterparts.

This all leads up to the stated possibility that such type of breakthrough might even unlock the mystery of how to divide a given angle of unknown size into three equal parts when acting upon it only by means of applying a *straightedge* and *compass*; thereby accomplishing the impossible feat of *Euclidean trisection*.

Naturally, on method which could be applied in order to achieve such goal would entail attempting to *geometrically construct* ***cube roots***. In this regard, the association that ***Equation sub-element theory*** bears upon such ***cube roots*** phenomenon is presented below wherein, as *algebraic* interpretations become supplied, they obviously would become disqualified as methods which could accomplish such *Euclidean trisection* feat.

a) Explaining why attempting to *geometrically construct cube roots* is synonymous with trisection, and therefore *cannot* be achieved solely by *Euclidean means*:

With regard to the factor $\cos(2\omega)$, as contained in the variable ℓ of the *Cubic Resolution Transform (CRT) presented* below, an association with ***cube roots*** can be established as follows *(Ref. Section 13.3)*:

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right) f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0 \qquad \textit{[Ref. Equation 38]}$$

Such that

$$\ell = 2f\cos(2\omega) \qquad \textit{[Ref. Figure 11]}$$

Where the formula for a *Binomial Expansion* of the *cube* of the *polynomial* $A \pm B$ is as follows:

$$(A \pm B)^3 = A^3 \pm 3A^2B + 3AB^2 \pm B^3$$

For the specific circumstance when:

$$A = \cos(2\omega)$$

$$B = i\sin(2\omega)$$

$$\begin{aligned}(A \pm B)^3 &= [\cos(2\omega)]^3 \pm 3[\cos(2\omega)]^2[i\sin(2\omega)] + 3[\cos(2\omega)][i\sin(2\omega)]^2 \pm [i\sin(2\omega)]^3 \\ &= \cos^3(2\omega) \pm 3[1-\sin^2(2\omega)][i\sin(2\omega)] - 3[\cos(2\omega)][1-\cos^2(2\omega)] \mp i\sin^3(2\omega) \\ &= [4\cos^3(2\omega) - 3\cos(2\omega)] \pm i[3\sin(2\omega) - 4\sin^3(2\omega)] \\ &= \cos(6\omega) \pm i\sin(6\omega)\end{aligned}$$

Taking the *cube root* of each side affords:

$$A + B = \cos(2\omega) + i\sin(2\omega) = \sqrt[3]{\cos(6\omega) + i\sin(6\omega)}$$

$$A - B = \cos(2\omega) - i\sin(2\omega) = \sqrt[3]{\cos(6\omega) - i\sin(6\omega)}$$

Such that by summing the two above equations,

$$2\cos(2\omega) = \sqrt[3]{\cos(6\omega) + i\sin(6\omega)} + \sqrt[3]{\cos(6\omega) - i\sin(6\omega)}$$

Now, upon letting ψ represent $\cos(6\omega)$, the following equality can be established,

$$\cos^2(6\omega) + \sin^2(6\omega) = 1$$

$$\psi^2 + \sin^2(6\omega) = 1$$

$$\sin(6\omega) = \sqrt{1-\psi^2}$$

Then, by substituting this result into the equation above, it can be shown that,

$$\begin{aligned}2\cos(2\omega) &= \sqrt[3]{\psi + i\sqrt{1-\psi^2}} + \sqrt[3]{\psi - i\sqrt{1-\psi^2}} \\ &= \sqrt[3]{\psi + i\sqrt{(-1)(\psi^2-1)}} + \sqrt[3]{\psi - i\sqrt{(-1)(\psi^2-1)}} \\ &= \sqrt[3]{\psi + i^2\sqrt{\psi^2-1)}} + \sqrt[3]{\psi - i^2\sqrt{\psi^2-1)}} \\ &= \sqrt[3]{\psi - \sqrt{\psi^2-1)}} + \sqrt[3]{\psi + \sqrt{\psi^2-1)}}\end{aligned}$$

Since real values for ψ exist within the range from -1 to +1, then the radical $\sqrt{\psi^2-1}$ must be *imaginary* or equal to zero. Hence, except for such latter case, each of the terms appearing under the two *cube root radicals* indicated above must be *complex numbers*. Now, since taking the *cube root* of a complex number is synonymous with representing its trisector in a Cartesian Coordinate System, it would appear to be impossible to *geometrically construct* it solely by *Euclidean means*.

b) Showing how *cube roots* can be eliminated through algebraic manipulation:

Except for certain *very rare instances (Ref. Section 20)*, an *unknown quantity z* may be represented as the *negative* ***cube root*** of the summation of *second*, *third* and *fourth terms* of a given *Generalized Cubic Equation* for $\alpha=1$ that becomes mathematically reorganized as follows:

$$\alpha z^3+\beta z^2+\gamma z+\delta=0 \quad \textit{[Ref. Equation 32]}$$

$$z^3+\beta z^2+\gamma z+\delta=0$$

$$z^3=-\beta z^2-\gamma z-\delta$$

$$=(-1)(\beta z^2+\gamma z+\delta)$$

$$=(-1)^3(\beta z^2+\gamma z+\delta)$$

$$z=-\sqrt[3]{\beta z^2+\gamma z+\delta}$$

Since such *2nd and 3rd terms* include the *unknown root, z,* its value is ***required aforehand*** in order to determine the value of the left-hand side of the above equation. Hence, such algebraic relationship cannot contribute towards attempting to trisect an angle solely by Euclidean means *(Ref. Section 19)*.

1) For rational values of z_R *and* ζ when R=1 and $\beta=0$:

Interposing rational values of $z_R=\tan\theta$ and $\zeta=\tan(3\theta)$ into the *3θ Cubic Equation* enables results to be obtained which thereafter could be *geometrically constructed*, as based upon such input. For example, when $z_R=1/3$ *(Ref. Section 19 Example)*:

$$z_R^{\,3} - 3\zeta z_R^{\,2} - 3z_R + \zeta = 0 \qquad \textit{[3θ Cubic Equation]}$$

$$(1/3)^3 - 3\zeta(1/3)^2 - 3(1/3) + \zeta = 0$$

$$1/27 + \zeta(1 - 1/3) - 1 = 0$$

$$\zeta(18/27) = 26/27$$

$$\zeta = 13/9$$

A *second, independent Generalized Cubic Equation (GCE)* for $R = 1$ and $\beta = 0$ can be determined as:

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3 + \gamma} \qquad \textit{[Ref. Equation 51]}$$

$$1/3 = \frac{3(13/9)(\gamma - 1) - 4(0)}{3 + \gamma}$$

$$(1/3)(3 + \gamma) = (13/3)(\gamma - 1)$$

$$3 + \gamma = 13\gamma - 13$$

$$16 = 12\gamma$$

$$4/3 = \gamma$$

Hence, the two above determined equations can be combined in order to be *resolved* simultaneously via the *Quadratic Formula*, or the ***geometric construction*** *Mapping Process* presented in *Section 2.3*, as follows:

$$z_R^{\,3} - 3\zeta z_R^{\,2} - 3z_R + \zeta = 0$$

$$z_R^{\,3} = 3\zeta z_R^{\,2} + 3z_R - \zeta$$

For $\alpha = 1$

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 + \beta z^2 + \gamma z + \delta = 0$$

Via substitution from above:

$$[3\zeta z_R^{\,2} + 3z_R - \zeta] + \beta z_R^{\,2} + \gamma z_R + \delta = 0$$

$$(3\zeta + \beta)z_R^{\,2} + (3 + \gamma)z_R + (\delta - \zeta) = 0$$

$$z_R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textit{[Ref. Quadratic Formula]}$$

$$z_R = \frac{-(3 + \gamma) \pm \sqrt{(3 + \gamma)^2 - 4(3\zeta + \beta)(\delta - \zeta)}}{2(3\zeta + \beta)}$$

$$= \frac{-(3 + 4/3) \pm \sqrt{(13/3)^2 - 4(13/3 + 0)[\zeta(1 - \gamma) + \beta - \zeta]}}{2(13/3 + 0)}$$

$$= \frac{-(13/3) \pm \sqrt{(169/9) + 4(13/3)(\zeta\gamma - \beta)]}}{2(13/3)}$$

$$= \frac{-(13/3) \pm \sqrt{(169/9) + 4(13/3)(13/9)(4/3)}}{26/3}$$

$$= \frac{-(13/3) \pm (1/3)\sqrt{169 + 4(13/3)(13)(4/3)}}{26/3}$$

$$= \frac{-13 \pm 13\sqrt{(9+16)/9}}{26}$$

$$= \frac{-1 \pm 5/3}{2}$$

$$= 1/3; -4/3$$

Accordingly:

- From a *given angle* $3\theta = 55.30484647^{o}$, $\zeta = \tan(3\theta) = 13/9$ can be ***geometrically constructed***
- From the synthesis of such *two equations*, a common root $z_R = \tan\theta = 1/3$ can be ***geometrically constructed*** using the Quadratic Equation expressed above via the mapping process stipulated in *Section 2.3*
- From such ***geometrically constructed*** length of $z_R = \tan\theta = 1/3$, an *angle* θ then can be ***geometrically constructed*** which is equal to 18.43494882^{o}, or exactly 1/3 the magnitude of such *given angle* $3\theta = 55.30484647^{o}$. Since such geometric construction relies upon the results of an algebraic analysis as *aforehand knowledge*, such process does not qualify as a valid ***Euclidean*** trisection

Above, notice that it is not necessary to extract a ***cube root*** in order to algebraically determine such solution.

2) For $\beta=\gamma=0$:

An associated analysis begins by examining the *Generalized Cubic Equation* for conditions when $\alpha = 1$ as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 = -(\beta z^2 + \gamma z + \delta)$$

$$z = -\sqrt[3]{\beta z^2 + \gamma z + \delta}$$

Notice above that in order to calculate a ***root z***, it first becomes necessary to extract the ***cube root*** of a value which is comprised of *multiples* and *mathematical combinations* of such unknown *quantity*.

However, this doesn't apply when $\beta = \gamma = 0$ as follows:

$$z^3 + (0)z^2 + (0)z + \delta = 0$$

$$z^3 + \delta = 0 \qquad \textit{[Ref. Section 13.5]}$$

Where,

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

$$= \frac{\delta - 0}{1 - 0}$$

$$= \delta$$

Via substitution:

$$z_R{}^3 + \delta = 0$$

$$z_R{}^3 + \zeta = 0$$

$$(R \tan \theta)^3 + \zeta = 0$$

When $R = 1$, the above equation then relates $\tan\theta$ to $\zeta = \tan(3\theta)$ where,

- $\zeta = \tan(3\theta)$ is a value which can be *geometrically constructed* from any ***given angle 3θ***
- $\tan\theta$ is a value from which ***trisected angle θ*** can be *geometrically constructed*

Under such conditions,

$$\tan^3 \theta + \zeta = 0$$

Via further substitution of *Equation 3*:

$$\tan^3\theta + \frac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta} = 0$$

$$\tan^2\theta + \frac{(3-\tan^2\theta)}{1-3\tan^2\theta} = 0$$

$$\tan^2\theta(1-3\tan^2\theta) + 3 - \tan^2\theta = 0$$

$$-3\tan^4\theta + 3 = 0$$

$$-\tan^4\theta + 1 = 0$$

$$1 = \tan^4\theta$$

$$\pm 1 = \tan^2\theta$$

$$\pm 1; i = \tan\theta$$

$$\pm 45^o = \theta$$

$$45^o; 315^o = \theta$$

$$135^o; 945^o = 3\theta$$

$$135^o; 225^o = 3\theta$$

$$\mp 1 = \zeta$$

Accordingly,

$$\tan^3\theta + \zeta = 0$$

$$\tan^3\theta \mp 1 = 0$$

$$\tan\theta = \pm\sqrt[3]{1}$$

$$\tan\theta = \pm 1$$

Since the ***cube root*** of *unity* is <u>defined</u> as *unity*, an algebraic solution becomes afforded without having to extract such ***cube root***.

This above *finding* is independently confirmed by *Equation 51* which applies because $z_R = R\tan\theta = (1)\tan\theta = \tan\theta$ as follows:

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

$$= \frac{3\zeta(0-1) - 4(0)}{3+0}$$

$$= -\zeta \qquad \textit{[Ref. Section 20.1]}$$

$$= {z_R}^3 \qquad \textit{[Since } {z_R}^3 + \zeta = 0 \textit{ above]}$$

$$1 = {z_R}^2$$

Taking the *square root* produces values for z_R as follows:

$$\sqrt{1} = z_R$$

$$\pm 1 = z_R \quad \textit{[Ref. Section 20.1]}$$

$$= \tan\theta$$

$$\arctan(\pm 1) = \theta$$

$$45^o;135^o = \theta$$

$$135^o;45^o = 3\theta$$

$$\tan 135^o;\tan 45^o = \tan(3\theta)$$

$$-1;+1 = \zeta$$

Check,

$$z_R^{\ 3} + \zeta = 0 \qquad z_R^{\ 3} + \zeta = 0$$

$$z_R^{\ 3} - 1 = 0 \qquad z_R^{\ 3} + 1 = 0$$

$$1^3 - 1 = 0 \qquad (-1)^3 + 1 = 0$$

$$1 - 1 = 0 \qquad -1 + 1 = 0$$

$$0 = 0 \qquad 0 = 0$$

As such, the two specifically determined *Generalized Cubic Equations*, $z_R^{\ 3} = \pm 1$, do not require ***cube roots*** to be *geometrically constructed* because they can be reduced to respective *Quadratic Equations* as demonstrated above.

3) **For Circumstances when *Generalized Cubic Equations* exhibit coefficients in prescribed ratios:**
By now, it should be realized that conducting *geometric construction* upon any *given value* of $\zeta = \tan(3\theta)$ is far different than *geometrically assessing* ***coefficients*** which belong to an *associated Cubic Equation*. Moreover, this distinction applies even when such *coefficients* just so happen to be ***irrational.***

This is because *algebraic assessment* and *geometric construction* are ***far different*** entities. So, it seems fitting that they, indeed, are represented by *different branches* of *mathematics*.

And so it is that trisection can be *algebraically* determined far more readily from given *cubic equations* than solely from *given geometric values* of $\zeta = \tan(3\theta)$; despite the fact that such algebraic solutions cannot constitute *Euclidean* trisections!

Algebraic determinations of such types become accomplished simply by first *interpreting*, and thereafter *geometrically operating* upon the *coefficient structures* of *given Cubic Equations*.

Perhaps the example which is easiest to comprehend pertains to a ***cubic root*** which, in fact, is equal to a fraction of a coefficient which appears in a *Generalized Cubic Equation*. For purposes of illustration, for:

$$\beta = -3z_R$$

$$-\frac{\beta}{3} = z_R$$

$$0 = z_R + \frac{\beta}{3}$$

The ***cube*** of the *above binomial* is:

$$0 = (z_R + \frac{\beta}{3})^3$$

$$= z_R{}^3 + 3(\beta/3)z_R{}^2 + 3(\beta/3)^2 z_R + (\beta/3)^3$$

$$= z_R{}^3 + \beta z_R{}^2 + (\beta^2/3)z_R + \beta^3/27$$

Such that,

$$0 = \alpha z^3 + \beta z^2 + \gamma z + \delta$$ *[Ref. Equation 32]*

Matching *like coefficients* renders:

$$\alpha = 1$$

$$\gamma = \beta^2/3$$

$$\delta = \beta^3/27$$

As such, a *Generalized Cubic Equation* whose ***coefficients*** appear in the *respective proportions* afforded below contains a *root* equal to $z_R = -\beta/3$:

$$z_R{}^3 + \beta z_R{}^2 + (\beta^2/3)z_R + \beta^3/27 = 0$$

Notice that for this above case, the value of the *coefficient* β can be either *rationally-based*, or *cubic irrational*.

The *geometric construction* aspect of this analysis becomes rudimentary since it consists simply of *geometric*ally dividing any given value of β into three equal portions in order to determine the value of its associated *root* z_R.

Moreover, since $\beta^2 = 3\alpha\gamma = 3(1)\gamma = 3\gamma$, the following equation also applies *(Ref. Section 13.2)*:

$$z_R = R\tan\theta = \frac{-\beta + \sqrt[3]{\beta^3 - 27\alpha^2\delta}}{3\alpha}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - 27(1)^2\delta}}{3(1)}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - 27(\beta^3/27)}}{3}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - \beta^3}}{3}$$
$$= \frac{-\beta}{3}$$

However, in many cases note that $R \neq 1$.

As indicated above, the ***cube root*** term always adds out to zero when making use of such *Generalized Cubic Equation* ***format***.

Check,

$$\tan(3\theta) = \zeta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \quad \textit{[Ref. Equation 3]}$$
$$= \frac{z_R(3 - z_R{}^2)}{1 - 3z_R{}^2}$$
$$= \frac{-\frac{\beta}{3}[3 - (-\frac{\beta}{3})^2]}{1 - 3(-\frac{\beta}{3})^2}$$
$$= \frac{-\frac{\beta}{3}(3 - \frac{\beta^2}{9})}{1 - (\frac{\beta^2}{3})}$$
$$= \frac{\frac{\beta^3}{27} - \beta}{1 - (\frac{\beta^2}{3})}$$
$$= \frac{\delta - \beta}{1 - \gamma} \quad \textit{[Ref. Equation 36]}$$

Hence, by comparing like aspects of the above two equations:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$ *[Ref. Equation 32]*

$$z_R^{\ 3} + \beta z_R^{\ 2} + (\beta^2/3)z_R + \beta^3/27 = 0$$ *Q.E.D.*

Unfortunately, this *above analysis* represents *little more* than determining equations for any ***prescribed root*** $\boldsymbol{z_R}$ whose *coefficient β can be acted upon via* **geometric construction** for purposes of <u>again</u> identifying or producing such ***given root***.

Three other *Cubic Equations* of the above *format* are determined below through a *simplified process*. One exhibits a *rational cubic root*, another contains a *cubic root* comprised of a *square root quantity* that can be *geometrically constructed* via the mapping process specified in *Section 2.3*, and another expresses an *cubic irrational cubic root* as follows:

For

$$z_R = \tan\theta = 1/5$$

$$\beta = -3z_R = -3/5$$

$$\gamma = \beta^2/3 = 3/25$$

$$\delta = \beta^3/27 = \gamma\beta/9 = -1/125$$

For

$$z_R = \tan\theta = 3 + \sqrt{7}$$

$$\beta = -3z_R = -3(3+\sqrt{7})$$

$$\gamma = \beta^2/3 = 3(16+6\sqrt{7})$$

$$\delta = \beta^3/27 = \gamma\beta/9 = -1(90+34\sqrt{7})$$

For

$$z_R = \tan\theta = \tan 20^o = 0.363970234$$

$$\beta = -3z_R = -1.091910703$$

$$\gamma = \beta^2/3 = 0.397422994$$

$$\delta = \beta^3/27 = \gamma\beta/9 = -0.048216713$$

Check,

$$z^3 - \frac{3}{5}z^2 + (\frac{3}{25})z - \frac{1}{125} = 0$$

$$(\frac{1}{5})^3 - \frac{3}{5}(\frac{1}{5})^2 + (\frac{3}{25})(\frac{1}{5}) - \frac{1}{125} = 0$$

$$1 - 3 + 3 - 1 = 0$$

$$0 = 0$$

$$z^3 - 3(3+\sqrt{7})z^2 + 3(16+6\sqrt{7})z - (90+34\sqrt{7}) = 0$$

$$(3+\sqrt{7})^3 - 3(3+\sqrt{7})(3+\sqrt{7})^2 + 3(16+6\sqrt{7})(3+\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(27+27\sqrt{7}+63+7\sqrt{7}) - 3(3+\sqrt{7})(16+6\sqrt{7}) + 3(16+6\sqrt{7})(3+\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(90+34\sqrt{7}) + (3-3)(3+\sqrt{7})(16+6\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(90+34\sqrt{7})-(90+34\sqrt{7})=0$$
$$0=0$$

$$z^3-1.091910703z^2+0.397422994z-0.048216713=0$$
$$(0.363970234)^3-1.091910703(0.363970234)^2+0.397422994(0.363970234)-0.048216713=0$$
$$0.048216713-0.14465014+0.14465014-0.048216713=0$$
$$0=0$$

From these above determined Cubic Equations, roots may be determined ***linearly*** via the expression *posed in Equation 51 as follows:*

For $z^3-\frac{3}{5}z^2+(\frac{3}{25})z-\frac{1}{125}=0$

Where,

$$\zeta=\frac{\delta-\beta}{1-\gamma}$$ *[Ref. Equation 36]*

$$=\frac{-\frac{1}{125}+\frac{3}{5}(\frac{25}{25})}{\frac{125}{125}-\frac{3}{25}(\frac{5}{5})}$$

$$=\frac{74}{110}$$

$$z_R=\frac{3\zeta(\gamma-1)-4\beta}{3+\gamma}$$ *[Ref. Equation 51]*

$$=\frac{3(\frac{74}{110})[\frac{3}{25}(\frac{5}{5})-(\frac{125}{125})]-4(-\frac{3}{5})(\frac{25}{25})}{3(\frac{125}{125})+\frac{3}{25}(\frac{5}{5})}$$

$$=\frac{3(\frac{74}{110})(-110)+300}{375+15}$$

$$=\frac{78}{390}$$

$$=\frac{1}{5}$$ *Q.E.D.*

For $z^3-3(3+\sqrt{7})z^2+3(16+6\sqrt{7})z-(90+34\sqrt{7})=0$

Where,

$$\zeta=\frac{\delta-\beta}{1-\gamma}$$ *[Ref. Equation 36]*

$$= \frac{-(90+34\sqrt{7})+3(3+\sqrt{7})}{1-3(16+6\sqrt{7})}$$

$$= \frac{81+31\sqrt{7}}{47+18\sqrt{7}}$$

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma}$$ *[Ref. Equation 51]*

$$= \frac{3(\frac{81+31\sqrt{7}}{47+18\sqrt{7}})[3(16+6\sqrt{7})-1]+12(3+\sqrt{7})}{3+3(16+6\sqrt{7})}$$

$$= \frac{(\frac{243+93\sqrt{7}}{47+18\sqrt{7}})(47+18\sqrt{7})+36+12\sqrt{7}}{51+18\sqrt{7}}$$

$$= \frac{279+105\sqrt{7}}{51+18\sqrt{7}}$$

$$= \frac{(3+\sqrt{7})(51+18\sqrt{7})}{51+18\sqrt{7}}$$

$$= 3+\sqrt{7}$$ *Q.E.D.*

For

$$z^3 - 1.091910703z^2 + 0.397422994z - 0.048216713 = 0$$

Where,

$$\zeta = \frac{\delta-\beta}{1-\gamma}$$ *[Ref. Equation 36]*

$$= \frac{-0.048216713+1.091910703}{1-0.397422994}$$

$$= \sqrt{3}$$

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma}$$ *[Ref. Equation 51]*

$$= \frac{3\sqrt{3}(0.397422994-1)+4(1.091910703)}{3+0.397422994}$$

$$= \frac{-3\sqrt{3}(0.602577005)+4.367642811}{3.397422994}$$

$$= \frac{-3.131081968+4.367642811}{3.397422994}$$

$$= \frac{1.236560843}{3.397422994}$$

$= 0.363970234$ *Q.E.D.*

4) For Applications of the Trisector Equation Generator:

Naturally it is of far greater interest to derive an ***algorithm*** which instead determines <u>equation types</u> from given, or known values of $\zeta = \tan(3\theta)$ where their associated ***cube root*** terms also *add out to zero*.

This is accomplished as follows, where:

$$z_R = R\tan\theta = (1)\tan\theta = \tan\theta$$

$$\alpha z_R^{\ 3} + \beta z_R^{\ 2} + \gamma z_R + \delta = 0$$ *[Ref. Equation 32]*

$$(1)\tan^3\theta + \beta\tan^2\theta + \gamma\tan\theta + \delta = 0$$ *(for α=1)*

Such that,

$$\zeta = \frac{\delta-\beta}{1-\gamma}$$ *[Ref. Equation 36]*

$$\zeta(1-\gamma)+\beta = \delta$$

$$\zeta(1-\frac{\beta^2}{3})+\beta = \delta$$ *(for $\gamma = \beta^2/3$)*

Substitution into what appears below gives:

$$\tan^3\theta + \beta\tan^2\theta + \gamma\tan\theta + \delta = 0$$

$$\tan^3\theta + \beta\tan^2\theta + (\frac{\beta^2}{3})\tan\theta + [\zeta(1-\frac{\beta^2}{3})+\beta] = 0$$

$$\frac{\beta^2}{3}(\tan\theta-\zeta) + \beta(1+\tan^2\theta) + \tan^3\theta + \zeta = 0$$

$$\beta^2 + [\frac{3(1+\tan^2\theta)}{(\tan\theta-\zeta)}]\beta + (3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)} = 0$$

Completing the square gives:

$$\beta^2+[\frac{3(1+\tan^2\theta)}{(\tan\theta-\zeta)}]\beta+[\frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}]^2+(3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)}=\frac{9(1+2\tan^2\theta+\tan^4\theta)}{4(\tan\theta-\zeta)^2}$$

$$[\beta+\frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}]^2+(3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)}(\frac{4}{4})(\frac{\tan\theta-\zeta}{\tan\theta-\zeta})=\frac{9(1+2\tan^2\theta+\tan^4\theta)}{4(\tan\theta-\zeta)^2}$$

Whereby,

$$[\beta+\frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}]^2=\frac{9(1+2\tan^2\theta+\tan^4\theta)}{4(\tan\theta-\zeta)^2}-(3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)}(\frac{4}{4})(\frac{\tan\theta-\zeta}{\tan\theta-\zeta})$$

$$\beta+\frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}=[\frac{1}{2(\tan\theta-\zeta)}]\sqrt{9(1+2\tan^2\theta+\tan^4\theta)-12(\tan^3\theta+\zeta)(\tan\theta-\zeta)}$$

$$\beta=[\frac{1}{2(\tan\theta-\zeta)}][-3(1+\tan^2\theta)\pm\sqrt{9(1+2\tan^2\theta+\tan^4\theta)-12(\tan^4\theta-\zeta\tan^3\theta+\zeta\tan\theta-\zeta^2}]$$

Equation 52. ***Trisector Equation Generator for z_R=-β/3.***

$$\beta=[\frac{1}{2(\tan\theta-\zeta)}][-3(1+\tan^2\theta)\pm\sqrt{9+12\zeta^2-12\zeta\tan\theta+18\tan^2\theta+12\zeta\tan^3\theta-3\tan^4\theta}]$$

Therefore, for any *postulated real value* of $\zeta=\tan(3\theta)$ and its *associated, calculated value* $z_R=R\tan\theta=(1)\tan\theta=\tan\theta$, the *coefficients* β, $\gamma=\beta^2/3$, and $\delta=\zeta(1-\frac{\beta^2}{3})+\beta$ can be calculated in order to describe a *Generalized Cubic Equation* whose *root* is $z_R=-\beta/3$.

For the case when:

$$\zeta=\tan(3\theta)=13/9$$
$$3\theta=55.30484647^o$$
$$\theta=18.43494882^o$$
$$z_R=\tan\theta=1/3$$

Then, by applying *Equation 52*:

$$\beta=[\frac{1}{2(\tan\theta-\zeta)}][-3(1+\tan^2\theta)\pm\sqrt{9+12\zeta^2-12\zeta\tan\theta+18\tan^2\theta+12\zeta\tan^3\theta-3\tan^4\theta}]$$

$$=[\frac{1}{2(-10/9)}][-3(10/9)\pm\sqrt{(729+2028)/81-468/81+162/81+52/81-3/81}]$$

$$=[\frac{-9}{20}][-\frac{30}{9}\pm\frac{1}{9}\sqrt{(729+2028)-468+162+52-3}]$$

$$=[\frac{-1}{20}][-30\pm\sqrt{2757-468+162+49}]$$

$$=\frac{3}{2}\mp\frac{1}{20}\sqrt{2500}$$

$$=\frac{3\mp5}{2}$$

$$=-1;+4$$

$$\gamma = \frac{\beta^2}{3}$$
$$= \frac{1}{3};\frac{16}{3}$$

$$\delta = \zeta(1-\gamma) + \beta$$
$$= \frac{13}{9}(1-\frac{1}{3}) - 1; \frac{13}{9}(1-\frac{16}{3}) + 4$$
$$= \frac{13}{9}(\frac{2}{3}) - \frac{27}{27}; \frac{13}{9}(-\frac{13}{3}) + 4(\frac{27}{27})$$
$$= \frac{1}{27};-\frac{61}{27}$$

Hence, such above determined *coefficients* generate the following pair of *Generalized Cubic Equations*:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$ *[Ref. Equation 32]*

$$z^3 - z^2 + \frac{1}{3}z - \frac{1}{27} = 0$$

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

Check,

For

$$z^3 - z^2 + \frac{1}{3}z - \frac{1}{27} = 0$$

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma}$$ *[Ref. Equation 51]*

$$= \frac{3(\frac{13}{9})[\frac{1}{3} - 1(\frac{3}{3})] - 4(-1)(\frac{9}{9})}{3(\frac{9}{9}) + \frac{1}{3}(\frac{3}{3})}$$

$$= \frac{(\frac{13}{3})(\frac{-2}{3}) + (\frac{36}{9})}{\frac{27}{9} + \frac{3}{9}}$$

$$= \frac{10}{30}$$

$$= \frac{1}{3}$$

For

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma}$$

$$= \frac{3(\frac{13}{9})[\frac{16}{3} - 1(\frac{3}{3})] - 4(4)(\frac{9}{9})}{3(\frac{9}{9}) + \frac{16}{3}(\frac{3}{3})}$$

$$= \frac{(\frac{13}{3})(\frac{13}{3}) - 4(4)(\frac{9}{9})}{\frac{27+48}{9}}$$

$$= \frac{25}{75}$$

$$= \frac{1}{3}$$

Also:

$$\beta^2 = 3\alpha\gamma \qquad \beta^2 = 3\alpha\gamma$$

$$= 3(1)(1/3) \qquad = 3(1)(16/3)$$

$$\beta = \pm\sqrt{1} \qquad \beta = \pm\sqrt{16}$$

$$\beta_2 = -1 \qquad \beta_1 = +4$$

So,

$$z_R = \frac{-\beta_2 + \sqrt[3]{\beta_2{}^3 - 27\alpha^2\delta}}{3\alpha} = \frac{+1+\sqrt[3]{(-1)^3 - 27(+1)^2(-\frac{1}{27})}}{3} = \frac{+1+\sqrt[3]{-1+1}}{3} = \frac{1+0}{3} = \frac{1}{3}$$

$$z_R = \frac{-\beta_1 + \sqrt[3]{\beta_1{}^3 - 27\alpha^2\delta}}{3\alpha} = \frac{-4+\sqrt[3]{(4)^3 - 27(1)^2(-\frac{61}{27})}}{3} = \frac{-4+\sqrt[3]{125}}{3} = \frac{-4+5}{3} = \frac{1}{3}$$

$$z^3 - z^2 + (1/3)z - 1/27 = 0$$

$$(\frac{1}{3})^3 - (\frac{1}{3})^2(\frac{3}{3}) + \frac{1}{3}(\frac{1}{3})(\frac{3}{3}) - \frac{1}{27} = 0$$

$$(1-3+3-1)/27 = 0$$

$$0 = 0$$

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

$$(\frac{1}{3})^3 + 4(\frac{1}{3})^2(\frac{3}{3}) + \frac{16}{3}(\frac{1}{3})(\frac{3}{3}) - \frac{61}{27} = 0$$

$$\frac{1+12+48-61}{27} = 0$$

$$0 = 0$$

Now with regard to these newly determined equations, The ***common root*** $z_R = 1/3$ for the first *given Cubic Equation* above can be *geometrically constructed* without having to take a *cube root* since such ***cube root term*** *adds out to zero.*

Moreover, such first *given Cubic Equation*, as cited above, contains $z_R = 1/3 = -\beta/3$ as a root; thereby represents the tangent of the *trisected angle θ*, the latter of which then could be *geometrically constructed* very easily.

With regards to the *second above given Cubic Equation*, $z^3 + 4z^2 + (16/3)z - 61/27 = 0$, its *associated root* z_R can *be geometrically constructed* from its *given coefficients* via application of *Equation 51*, as shown above. Hence, in this particular case, it also is not necessary to obtain a ***cube root*** via *geometric construction*.

For such two *given Cubic* Equations, as are represented above, the following proof is provided in order to demonstrate that each relate to the same *angle* $3\theta = 55.30484647^o$:

For

$$z^3 - z^2 + \frac{1}{3}z - \frac{1}{27} = 0$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

$$= \frac{-\frac{1}{27} - (-1)(\frac{27}{27})}{1(\frac{27}{27}) - \frac{1}{3}(\frac{9}{9})}$$

$$= \frac{-1 + 27}{27 - 9}$$

$$= 26/18$$

$$\tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^o$$

For

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$

$$= \frac{-\frac{61}{27} - (4)(\frac{27}{27})}{1(\frac{27}{27}) - \frac{16}{3}(\frac{9}{9})}$$

$$= \frac{-61 - 108}{27 - 144}$$

$$= -169/-117$$

$$\tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^o$$

Therefore, a *given angle* of $3\theta = 55.30484647^o$ can be ***divided*** *into three equal angles* of $\theta = 55.30484647^o / 3 = 18.43494882^o$ each by means of a *geometric construction* which utilizes nothing more than a *straightedge* and *compass* when ***applying*** the *coefficients* and *respective formats* expressed in *either* of the above determined *Cubic Equations*.

In conclusion, ***Generalized Cubic Equation formats*** exhibiting a ***sub-element*** *of* $R=1$ contain a *root* of $z_R = \tan\theta$ with respect to their characteristic values of $\zeta = \tan(3\theta)$ such that,

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

Such values z_R and ζ can be determined by a *geometric construction* which employs *only straightedge and compass* instruments that operate solely upon *various inherent coefficients* resident within these *formats*.

Since an ***angle of 3θ*** can be ***geometrically constructed*** from a *given* value of $\zeta = \tan(3\theta)$, and since an ***angle of θ*** also can be ***geometrically constructed*** from such previously *algebraically determined* value of $z_R = \tan\theta$, trisection can be *achieved* through *geometric manipulation* of such *inherent coefficients*.

This does not constitute a bonafide *Euclidean Trisection* event, however, since such ***Generalized Cubic Equation formats*** exist merely as algebraic *transformations* that constitute ***aforehand knowledge*** of such *desirable root structures* in the first place *(Ref. Section 19)*.

SECTION 22

(In the event of any conflict between this section and U.S. Patent No. 10994569 issued on 5/4/2021, the latter governs)

Astounding as it might sound, *cubic irrational lengths* actually can be ***depicted*** from any *arbitrarily assigned* or *given* length of unity *without* having to defy or otherwise violate the *conclusion* expressed in *Section 9.1*.

This is achieved by a process whereby *cubic irrational lengths* become ***geometrically formed*** instead of *geometrically constructed*!

Such process furthermore enables *trisected angles*, respectively equal to exactly one-third the magnitude of any given angles, now also to become *portrayed*.

With respect to the above, the prospect of identifying *cubic irrational lengths* is considered to be of ***far greater importance*** than actually *trisecting* various ascribed angles of 3θ.

> This is because the concept of depicting *exact cubic irrational lengths* alongside an amalgamation of *rationally-based lengths* that actually define them should exemplify a fitting or fundamentally new ***Number Theory*** groundwork, in itself, from which to launch amazing new discovery; and thereby, further advance the overall state-of-the-art!
>
> In contrast, ***trisecting an angle*** from a given angle 3θ, although of ***significant import***, nevertheless does not appear to possess the same *profound capability* to stand alone as an actual *groundwork* in itself, from which to derive other meaningful applications.

Section 22.1 indicates that ***geometrically formed cubic irrational lengths*** become evident during ***overlapment***, a singular condition observed to occur whenever the *longitudinal axis* of a pre-selected *compass arm* (belonging to a new appurtenance consisting *solely* of *compass* and *straightedges* interconnected in a unique manner) hovers *directly over* the *determinable point* (η,τ).

Cubic irrational lengths result because ***geometric constraint*** becomes imposed upon the *endpoint* of the other compass arm.

Setting all *compass arm* and *straightedge* lengths equal to an arbitrary value of *unity* assures that resulting *cubic irrational lengths* can become depicted directly alongside such *rational unitary basis*.

Section 22.2 stigmatizes *Euclidean practice* as being somewhat ***incomplete***, primarily because it *lacks* the capability to ***geometrically construct*** *cubic irrational lengths.*

In order to remedy this inadequacy, an examination of *intersection points* that occur at various locations along straight lines, rather than only at their terminations, or *endpoints* was undertaken.

This surfaced additional *intersection points* not normally encountered during generally accepted *Euclidean practice*.

Why such investigations were not conducted earlier, say by Euclid and his crew, is subject to controversy; but two possibilities exist which may be attributed to either:

a) **Oversight:** Whereby such *intersection points* were overlooked; that is, they simply went undetected along the way; or

b) **Mathematical indifference:** Whereby such *intersection points* were deliberately ignored during prior exercises because identifying midway locations in such manner then might have been considered to be outside the scope of the very rules, regulations, and interpretations which govern ***geometric construction*** via *Euclidean compass* and *straightedge* tools.

In either event, generally accepted *Euclidean practice* presently remains limited in that it can ***geometrically construct*** only *rationally-based lengths*.

Had *Euclid* and his contemporaries been advised that *cubic irrational lengths* actually could be depicted solely from a unique arrangement of *compasses* interconnected via *straightedge*, such capability most definitely would have been incorporated into their practice long ago.

Section 22.3 highlights various aspects of what is, and what should be acknowledged to be, generally accepted *Euclidean practice* as follows:

- ***Section 22.3.1*** affords examples of *relative motion* evidenced within generally accepted *Euclidean practice*.
- ***Section 22.3.2*** gives an example of an imposition of *geometric constraint* exhibited by the generally accepted *Euclidean practice* of tightening a compass hinge
- ***Section 22.3.3*** asserts that because *intersection points* can be determined via ***geometric construction***, ***overlapment*** should be categorized under the *Euclidean umbrella* since it too locates *intersection points*.

The only ***difference*** is that ***overlapment*** seeks to identify additional *intersection points* that previously were not determined by ***geometric construction***.

From the distant vantage point of Earth, such distinct *longitudinal axis* (previously mentioned in connection with ***overlapment***), once contemplated to exist outside of the realm of such aforementioned appurtenance *(Ref. Section 22.1)*, may be perceived as a straight line of seemingly imperceptible width which becomes drawn, for example, through *Orien's Belt*. At the precise moment when it is observed to pass either directly in front of or behind a particular star, no matter how faint, ***overlapment*** occurs at the specific location where such straight line is viewed to cross, or intersect *with* the star.

Such process also may be likened to a *total eclipse* of the sun by the moon. During this occurrence, a straight line fictitiously can be drawn which is considered to *intersect*:

- The center of the moon
- The center of the sun
- The midway point between the viewer's eyes

Hence, ***overlapment*** coexists with *intersection*. They go hand-in-hand, whereby at times they even might be perceived as being *inextricably linked* or associated to one another.

Section 22.4 recommends that generally accepted *Euclidean practice* becomes amended in order to hereby include the following stipulation:

The prospect of incorporating ***cubic irrational length*** *depictions* into formerly established *Euclidean practice* without violating, detracting from, or otherwise conflicting with its precepts *theoretically* would entail:

- Using only *Euclidean compass* and *straightedge* instruments in a manner entirely consistent with all of the rules and regulations applied during Euclid's day
- Treating ***cubic irrational length geometrically formed*** *depiction* in exactly the same manner as ***rationally-based geometric construction***; whereby both become determinable entirely from a given length of unity *(Ref. Section 9.1)*
- Acknowledging the *process* of obtaining ***geometrically formed*** depictions as a new Euclidean ***enhancement***; one which remains completely independent, or is distinguished entirely apart from the presently accepted *Euclidean process* of ***geometric construction***

By recognizing ***overlapment***, geometry then would become ***complete***; thereby identifying all possible *intersection points* associated with a given length of *unity (Ref. Section. 9.1 Conclusion)*.

It also then would enable exact depictions of both *rationally-based* and *cubic irrational lengths* alongside one another!

Section 22.5 presents the associated theory which enables *cubic irrational length depictions*, recapped as follows:

Cubic irrational numbers are known to manifest themselves as *cubic root values* z_R, z_S, and z_T inherent within ***3θ Cubic Equation****s*.

In consonance with the *Cubic Equation Cubic irrational Root Uniqueness Theorem*, this may be interpreted to mean *(Ref. Section 9.3)*:

When a ***3θ Cubic Equation***, of the particular form designated below, possesses a *rationally-based coefficient* of $\zeta = \tan(3\theta)$, its *roots* nevertheless still may be *cubic irrational*.

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

During such circumstances, a ***mutual existence*** between equation *rationally-based coefficients* and *associated cubic irrational roots* presumably occurs.

Table 35 charts examples of *cubic irrational lengths* stemming from the *3θ Cubic Equation* for the two specific conditions when:

1) $\zeta = \tan(3\theta) = \sqrt{3}$; and
2) $\zeta = \tan(3\theta) = (3/8)\sqrt{57}$.

Table 35 relates how *cubic irrational root length values* ascertained from such specific *rationally-based values* become commissioned as actual ζ values in themselves, in order to perpetuate *numerical length determinations*.

Quite obviously, other *trigonometric depictions* besides those specified in *Error! Reference source not found.* can be determined as *offshoots* to such tangent determinations -- including both sine and cosine portrayals.

Even though ***all*** cubic irrational lengths (such as the value for $\pi = 3.14159265358979323846264338327 9...$) quite possibly cannot yet directly be ascertained via this above process, nevertheless it still significantly and sufficiently contributes to the ***overall advancement of Number Theory***, simply because it now equips humanity with a brand new, profound capability to actually depict *cubic irrational numbers* ***geometrically*** *(Ref. Related Problem Number 48)*!

Section 22.6 introduces **Atacins**, a novel invention with capability to depict a ***geometrically formed*** angle exactly one-third the magnitude of any *given angle* that becomes programmed into it.

Even when the tangent of such *resulting angle* is a *cubic irrational length*, *Atacins* depicts it. **Atacins** is an *acronym* for ***a****ngle* ***t****risector* ***a****nd* ***c****ubic* ***i****rrational length* ***i****nstrument*, whereby a motion must be *imparted* during such determinations.

The device *overcomes* the *rational number* to *cubic irrational number quandary* normally experienced during prior attempts at *Euclidean trisection*.

This is achieved by articulating such invention until ***overlapment***, as described above, occurs; whereby, *cubic irrational length*s become ***depicted*** alongside *given rationally-based* ones.

- ***Section 22.6.1*** indicates that such articulation, compass endpoint A′ is to be constrained within the *slot arrangement* appearing in compass arm $\overline{OA}$, thereby permitting it to *ride* only in the horizontal direction, or *actuate* only along the x-axis *(Ref. Figure 51)*.

  Atacins features *straightedge member* $\overline{OO'}$ whose *endpoints* interconnect to two hinges which belong to compasses OAB and O′A′B′, respectively *(Ref. Figure 51).*

  Therein, *members* $\overline{AB}$ and $\overline{A'B'}$ *extended* have been inserted only to replace the *tightening capabilities* of such respective compass hinges. Such modification simplifies the operation of the device, but is not mandatory.

  Accordingly, *Atacins* consists of *two hinges which* attach the endpoints of a *middle straightedge* to respective *assemblies* of *swinging arms* which collectively may be actuated as *independent compasses.*

  Specifically, *Atacins* is a mechanism that consists of a *middle straightedge member* that *interconnects* with two *independent assemblies* comprised of *identically shaped isosceles triangles*. Moreover, the assigned length of such *middle straightedge* is equal to that applied to each equal side of both isosceles triangles.

  - A detail for such *identically shaped isosceles triangles* appears in *Figure 50*
  - An embodiment of the device is presented in *Figure 51*

  For each *isosceles triangle*, *enclosed angles* located adjacent to such *middle straightedge member* are to be adjusted to a *known* or *given angle* $(90-3\theta)^{\circ}$.

- ***Section 22.6.2*** contributes an *overall proof* which validates that *trisection* becomes achieved for any and all magnitudes of $(90-3\theta)^{o}$

- ***Section 22.6.3*** presents two alternate *Atacins* configurations as follows:
    - The *first* replaces two existing arms with one new member *(Ref.* ***Section 22.6.3.1****)*
    - The *second* performs as an intricate *parallelogram* *(Ref.* ***Section 22.6.3.2****)*
    - The *third* (*Ref. Figures 32–59*) operates in much the same way as would a *car jack (Ref.* ***Section 22.6.3.3****)*

- ***Section 22.6.4*** states that *Atacins* depicts ***exact*** *cubic irrational lengths,* characterized by *decimal sequences* which are considered to *continue on indefinitely*, instead of repeating themselves.

    Such capability renders former approximation techniques, like the one described below, obsolete:

    Dividing up a given length of unity into ten equal portions (Ref. ***Error! Reference source not found.****), and then into hundredths (Ref.* ***Error! Reference source not found.****), and so on, until such desired cubic irrational length becomes amply gauged via ruler.*

Section 22.7 is a *summary* for the entire section. It mentions that *Atacins*:

- Enhances upon the *uncontrolled movement* allowed in former *Archimedes* geometric renderings by launching actuations exclusively from completely ***identifiable locations***. No *guess work* is required!

- Depicts *rationally-based lengths* directly alongside associated *cubic irrational root lengths.*

    Relates *rationally-based* coefficients to *cubic irrational root* counterparts.

- Enables the *trisector* of *any given angle* to be ***geometrically formed*** simply by applying the following two step process *(Ref. Figure 51)*:
    1) Set *angles AOB and A'O'B'* to *predetermined* angles of *90–3θ degrees* each;
    2) Then articulate, or flex the invention until such time that the *longitudinal axis* of *member* $\overline{O'B'}$ ***overlaps*** *point B*.

    The trisected *angle OO'C* thereafter becomes easily identified by *bisecting* the ***geometrically formed*** *angle OO'A'* either by use of added pencil/paper or via ruler.

There is no need to change the wording of the *conclusion* to *Section 9.1* because of the logic presented below:

- ***Rationally-based numbers*** comprise all *real numbers* which can be ***geometrically constructed*** from a given, arbitrary length of unity
- ***Cubic irrational numbers*** comprise all other *real numbers*; specifically, those which cannot be ***geometrically constructed*** from a given, arbitrary length of unity – which includes all those which can be ***geometrically formed*** from a given, arbitrary length of unity

SECTION 23

This final portion of the treatise delves into wave propagation.

Wave fronts can be depicted as curve snapshots over time. That is to say, as waves move, different curves can map them. The benefit of ***Equation Sub-element Theory*** is that it avails families of curves which, at times, can trace such propagation. Figure 60 gives an example of this such that:

> The moving wave portrayed at time t = 0 in *Figure 60* changes shape as it travels through a medium. For this particular wave, points A and B remain *stationary* as the wave disintegrates from time t= 0 to time $t = t_2$. However, node O located in the middle of the symmetrical wave travels through point O’ to a location of O’’ at time $t = t_2$. This presents an indication either that:
>
> - Weaker resisting wave forces are at play at the vertical plane that node O passes through as it moves from Node O’ to Node O’’ than those tending to resist points A and B from propagating during this same time period, or
> - Weaker applied *thermodynamic forces* reside at endpoints A and B of the moving wave than at its apex, or
> - Any combination thereof
>
> The term *plane* mentioned above applies to the fact that moving waves, such as that represented in *Figure 60*, assume three dimensional shapes in the real world. Their respective cross sections may be circular, elliptical, or any other variation that conceptually may be modeled over time. Furthermore, such cross sections may change shape affording additional provision in which to characterize the forces at work.

Another example of wave propagation is afforded which pertains to football players as they run a play. Such analysis comes complete with accompanying animations.

SECTION 24

Therein, various problems are analyzed. They are presented in the same sequence as theory is rendered in the body of treatise; thereby allowing for easy cross-referencing.